# Chapter 6

# The Discrete Fourier Transform

**Goal**: Given a finite-duration sequence $x[n]$, $0 \leq n \leq N - 1$, we want to obtain a Fourier representation for $x[n]$. This representation, known as the Discrete Fourier Transform (DFT), will enable us to analyze important properties for Linear systems. The DFT will be shown to be a very important tool in digital signal processing.

<u>Definition</u>: Given a finite-duration sequence $x[n]$, $0 \leq n \leq N - 1$, the DFT $X[k]$, $0 \leq k \leq N - 1$ is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}, \ 0 \leq k \leq N - 1 \tag{6.1}$$

The original sequence $x[n]$ can be obtained from the DFT representation $X[k]$ by the following

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N}, \ 0 \leq n \leq N - 1 \ \diamond \tag{6.2}$$

In these notes we will examine various properties pertaining to 6.1 and 6.2. An important property is the fact that $X[k]$ in 6.1 is periodic with period $N$, i.e.,

$$X[k + N] = X[k] \tag{6.3}$$

Also, in spite of the fact that $x[n]$ is of finite duration, $0 \leq n \leq N - 1$, we can still define $x[n]$ for values of $n > N$. As an example,
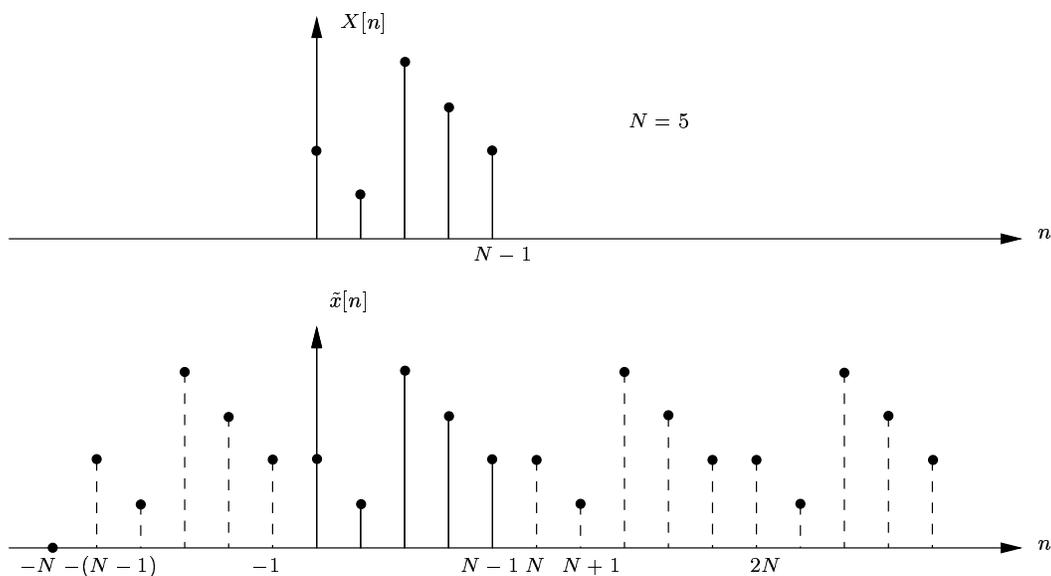
$$x[N + 1] = x[1] \tag{6.4}$$

Figure 6.1: A finite sequence $x[n]$ and its periodic extension $\tilde{x}[n]$.

This issue of periodicity can be best studied through the point of view that $x[n]$ is one period of a periodic sequence $\tilde{x}[n]$, and that $X[k]$ is one period of a periodic sequence $\tilde{X}[k]$. For example, figure 6.1 shows a finite sequence $x[n]$ of length 5 and its periodic extension $\tilde{x}[n]$.

Therefore, we will examine first the Fourier representation for periodic discrete sequences, the Discrete Fourier series (DFS).

## 6.1    The Discrete Fourier Series (DFS)

Consider a periodic sequence $x[n]$,

$$\tilde{x}[n] = \tilde{x}[n + rN] \tag{6.5}$$

where $r$ is an integer. We can represent $\tilde{x}[n]$ in terms of the complex exponentials $e^{j2\pi kn/N}$ as follows:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k] e^{j2\pi kn/N} \tag{6.6}$$

Since the set $e^{j2\pi kn/N}$ is periodic with period $N$, that is,

$$e^{\frac{j2\pi kn}{N}} = e^{\frac{j2\pi(k+rN)n}{N}} \tag{6.7}$$

the set $e^{j2\pi kn/N}$, $0 \le k \le N-1$ is complete. Therefore, we need only $N$ complex exponentials to completely represent $\tilde{x}[n]$. Hence, 6.6 can be written in the following finite summation:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j2\pi kn/N} \tag{6.8}$$

**Theorem**:

The coefficients $\tilde{X}[k]$, $0 \le k \le N-1$ in 6.8 can be obtained from the sequence $\tilde{x}[n]$ by the following summation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi kn/N} \tag{6.9}$$

**Proof**:

Multiply both sides of 6.8 by $e^{-j2\pi rn/N}$ and perform the summation from $n = 0$ to $N - 1$, that is,

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi rn/N} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j2\pi(k-r)n/N} \tag{6.10}$$

Interchange the order of summation in 6.10,

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi rn/N} = \sum_{k=0}^{N-1} \tilde{X}[k]\left\{\frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi(k-r)n/N}\right\} \tag{6.11}$$

Now the term between the braces has a value of one if $(k - r)$ is an integer multiple of the period $N$, i.e.,

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi(k-r)n/N} = \begin{cases} 1, & (k-r) = mN \\ 0, & \text{otherwise} \end{cases} \tag{6.12}$$

where $m$ is an integer. We need only consider the case $m = 0$, i.e.,

$$\frac{1}{N}\sum_{n=0}^{N-1}e^{j2\pi(k-r)n/N} = \begin{cases} 1, & k = r \\ 0, & \text{elsewhere} \end{cases} \tag{6.13}$$

From 6.13 and 6.11

$$\sum_{n=0}^{N-1}\tilde{x}[n]e-j2\pi rn/N = \sum_{k=0,k=r}^{N-1}\tilde{X}[k] \tag{6.14}$$

Therefore,

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi kn/N} \tag{6.15}$$

Q.E.D.

In summary, given a periodic sequence $\tilde{x}[n] = \tilde{x}[n+rN]$, $r$ is an integer, the Discrete Fourier Series (DFS) representation is:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j2\pi kn/N} \tag{6.16}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi kn/N} \tag{6.17}$$

If we define the parameter $W_N \triangleq e^{-j2\pi/N}$, we can write 6.16 and 6.17, in terms of $W_N$ as:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} \tag{6.18}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{-kn} \tag{6.19}$$

**Notes:**

1.     Both $\tilde{x}[n]$ and $\tilde{X}[k]$ are periodic with period $N$.

2.     $\tilde{X}[k]$ is, in general, complex. Therefore, it will be characterized in terms of an amplitude and phase for every value of $k$.

**Example 1:**

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty}\delta[n+rN] \tag{6.20}$$

where

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{6.21}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi kn/N} \tag{6.22}$$

$$= \sum_{n=0}^{N-1} \delta[n]e^{-j2\pi kn/N} \tag{6.23}$$

(Notice that over one period $\tilde{x}[n] = \delta[n]$)

$$\tilde{X}[k] = 1 \text{ for all } 0 \le k \le N-1 \tag{6.24}$$
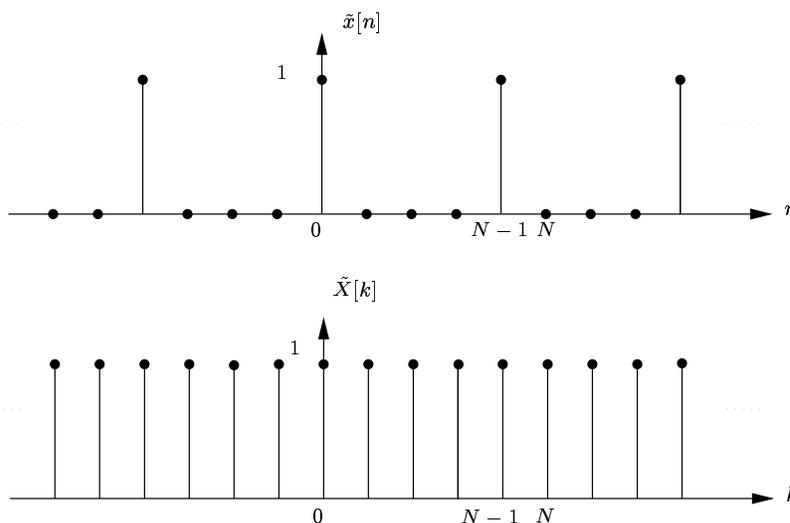
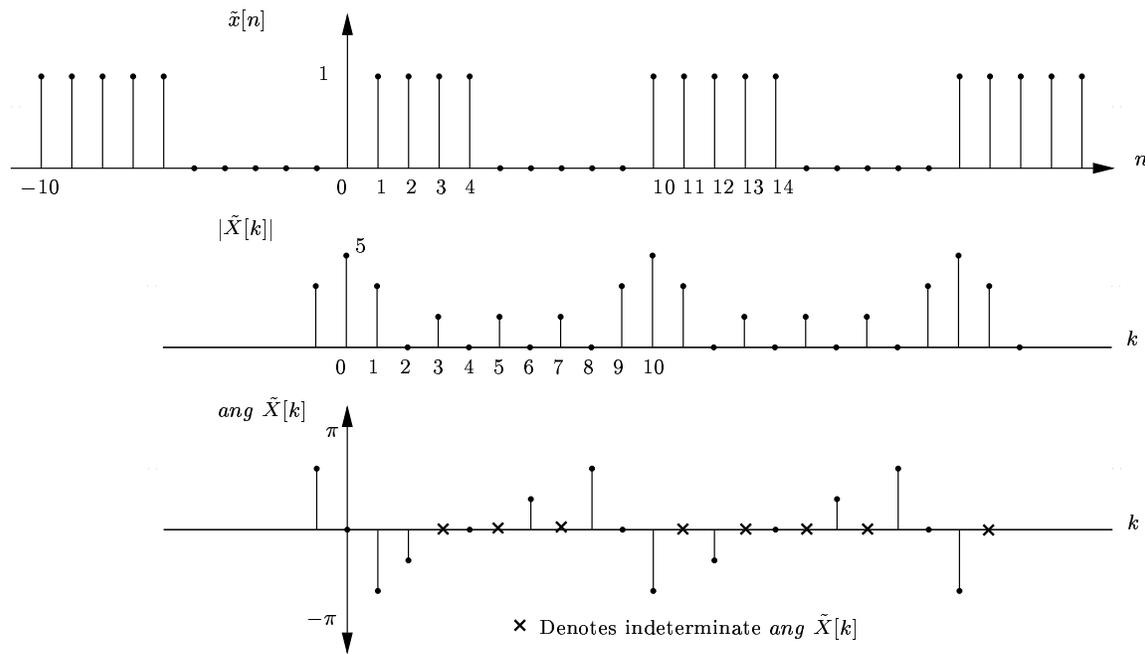The results are plotted in figure 6.2.

Figure 6.2: Plots for Example 1.

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**Example 2**:

$$\tilde{x}[n] \;=\; \sum_{r=-\infty}^{\infty} x[n+rN] \tag{6.25}$$

$$\text{where } x[n] \;=\; \begin{cases} 1, & o \le n \le 4 \\ 0, & \text{else} \end{cases} \tag{6.26}$$

$$\text{and } N \;=\; 10$$

$$\tilde{X}[k] \;=\; \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi k n/N}, \;\; N = 10$$

$$\;=\; \sum_{n=0}^{9} \tilde{x}[n] e^{-j2\pi k n/10} = \sum_{n=0}^{4} 1 \; e^{-j2\pi k n/10}$$

$$\;=\; \frac{1 - e^{-j2\pi k 5/10}}{1 - e^{-j2\pi k/10}} = e^{-j2\pi k/10} \frac{\sin\left(\pi k/2\right)}{\sin\left(\pi k/10\right)} \tag{6.27}$$



× Denotes indeterminate $ang\ \tilde{X}[k]$

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Relationship between the DFS representation $\tilde{X}[k]$ and the sequence Fourier transform $X(e^{jf_d})$ over one period of the periodic sequence $\tilde{x}[n]$. Let $x[n]$ represent one period of $\tilde{x}[n]$, i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \tag{6.28}$$
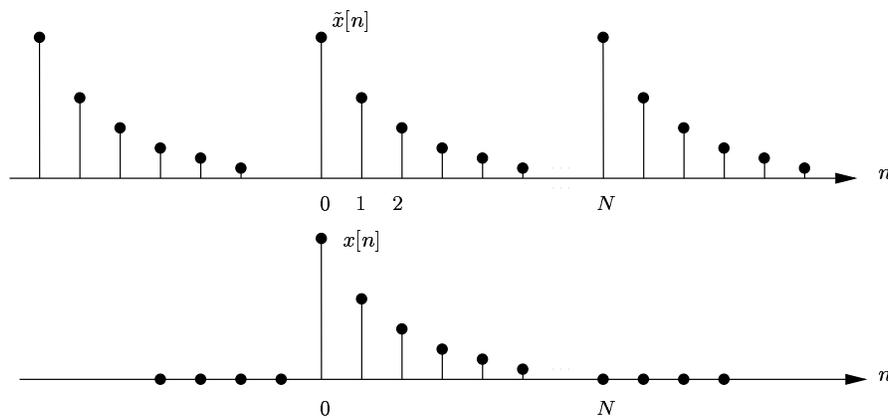
**Illustration**:



Figure 6.3:

Now,

$$X(f_d) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi f_d n} \tag{6.29}$$

$$\equiv \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi f_d n}$$

Therefore,

$$\tilde{X}[k] = \sum_{n=0}^{N-1} e^{-j2\pi kn/N} \tag{6.30}$$

Comparing 6.29 and 6.30, we see that

$$\tilde{X}[k] = X(f_d)|_{f_d = \frac{k}{N}} \tag{6.31}$$

This corresponds to sampling the sequence Fourier transform at $N$ equally spaced frequencies between $f_d = 0$ and $f_d = 1$ with a frequency spacing of $\frac{1}{N}$.

**Example 3**:

Consider $x[n]$ in example 2, i.e.,

$$x[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise} \end{cases} \tag{6.32}$$

The Fourier transform of one period $\tilde{x}[n]$ is given by

$$X(f_d) = \sum_{n=0}^{4} 1 \, e^{j2\pi f_d n} \tag{6.33}$$

$$= e^{-j4\pi f_d} \frac{\sin(5\pi f_d)}{\sin(\pi f_d)} \tag{6.34}$$

Comparing 6.27 with 6.34 we see that

$$\tilde{X}[k] = X(f_d)\big|_{f_d = \frac{k}{10}} \tag{6.35}$$

Recall:

      1.    $X(f_d)$ is continuous in $f_d$ and periodic with period 1.

      2.    $\tilde{X}[k]$ is discrete (defined only for integer values of $k$) and periodic with period $N$.

From 6.34 the magnitude and phase for $X(f_d)$ and $\tilde{X}[k]$ can be plotted as in figure 6.4.
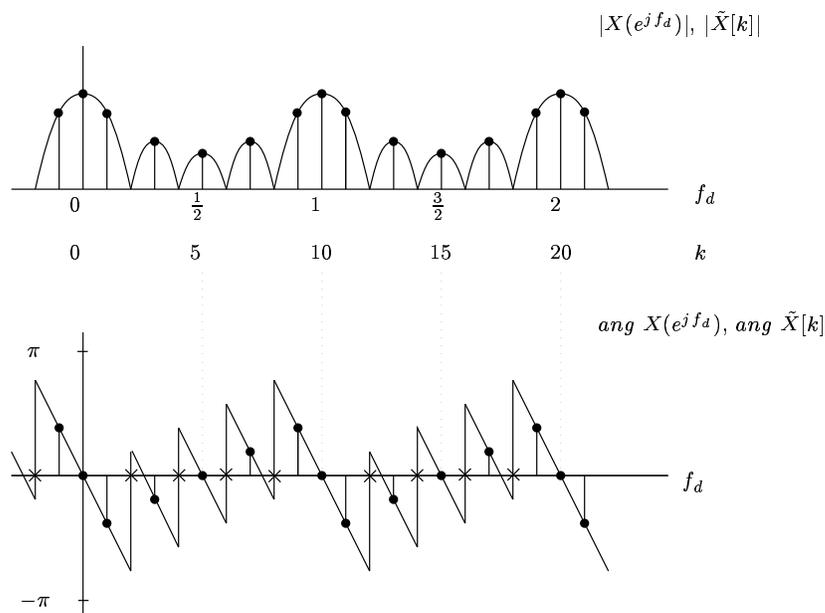
Figure 6.4: Plots for Example 3 illustrating that the DFS coefficients of a periodic sequence are samples of the sequence Fourier transform of one period.

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## 6.2 Properties of the Discrete Fourier Series (DFS)

(i) LINEARITY
Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period $N$ such that

$$\tilde{x}_1[n] \quad \overset{DFS}{\longleftrightarrow} \quad \tilde{X}_1[k] \tag{6.36}$$

$$\text{and } \tilde{x}_1[n] \quad \overset{DFS}{\longleftrightarrow} \quad \tilde{X}_1[k] \tag{6.37}$$

$$\text{Then } a\tilde{x}_1[n] + b\tilde{x}_2[n] \quad \overset{DFS}{\longleftrightarrow} \quad a\tilde{X}_1[k] + b\tilde{X}_2[k] \tag{6.38}$$

**Proof**: Use definitions.

(ii) SHIFT OF A SEQUENCE

   a. Time Shift

$$\text{If } \tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k] \tag{6.39}$$

$$\text{then } \tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} e^{-j2\pi km/N}\tilde{X}[k] \tag{6.40}$$

**Proof**:

$$\text{DFS}\{\tilde{x}[n-m]\} \overset{\triangle}{=} \sum_{n=0}^{N-1} \tilde{x}[n-m]e^{-j2\pi kn/N} \tag{6.41}$$

Let $[n-m] = l$,

$$\text{then } n\big|_{n=0}^{N-1}, \ l\big|_{-m}^{N-1-m}, \ n = l+m \tag{6.42}$$

$$\text{DFS}\{\tilde{x}[n-m]\} = \sum_{l=-m}^{N-1-m} \tilde{x}[n]e^{-j2\pi k(l+m)/N} \tag{6.43}$$

$$= e^{-j2\pi km/N}\sum_{l=-m}^{N-1-m} \tilde{x}[l]e^{-j2\pi kl/N} \tag{6.44}$$

$$\equiv e^{-j2\pi km/N}\tilde{X}[k] \tag{6.45}$$

Q.E.D.

   b. Frequency Shift

$$\text{If } \tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k] \tag{6.46}$$

$$\text{then } e^{j2\pi nl/N}\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k-l] \tag{6.47}$$

**Proof**:

$$\text{DFS}\{e^{j2\pi nl/N}\tilde{x}[n]\} \overset{\triangle}{=} \sum_{n=0}^{N-1} e^{j2\pi nl/N}\tilde{x}[n]e^{-j2\pi kn/N} \tag{6.48}$$

$$= \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j2\pi(k-l)n/N} \tag{6.49}$$

$$\equiv \tilde{X}[k-l] \tag{6.50}$$

Q.E.D.

(iii) DUALITY OF DISCRETE FOURIER SERIES COEFFICIENTS

Recall that for a continuous-time signal $x(t)$, the duality property of the Fourier transform was stated as follows:

$$\text{If } x(t) \quad \longleftrightarrow \quad X(f) \text{ then} \tag{6.51}$$

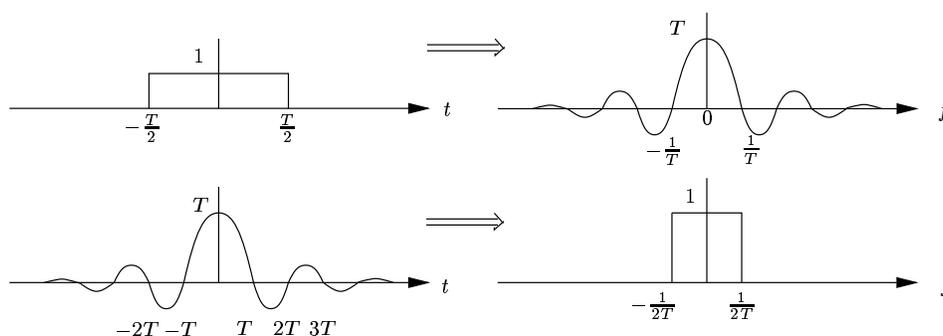$$X(t) \quad \longleftrightarrow \quad x(-f), \; f \text{ in Hz.} \tag{6.52}$$

**Example 4**:



Figure 6.5: Duality in the continuous-time.

Now consider the Discrete Fourier Series (DFS).

$$\tilde{x}[n] \quad \triangleq \quad \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}$$

$$\tilde{x}[-n] \quad = \quad \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{-j2\pi kn/N}$$

$$N\tilde{x}[-n] \quad = \quad \sum_{k=0}^{N-1} \tilde{X}[k] e^{-j2\pi kn/N}$$

$$\triangleq \quad \text{DFS}\left\{\tilde{X}[k]\right\}$$

Hence,

$$\text{If } \tilde{x}[n] \quad \overset{DFS}{\longleftrightarrow} \quad \tilde{X}[k] \text{ then} \tag{6.53}$$

$$\tilde{X}[n] \quad \overset{DFS}{\longleftrightarrow} \quad N\tilde{x}[-k] \tag{6.54}$$

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(iv) SYMMETRY PROPERTIES FOR THE DFS

Consider a *real* sequence $\tilde{x}[n]$, it is straight forward to prove the following properties:

a.  $\tilde{X}[k] = \overline{\tilde{X}}[-k]$

b.  $\mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\}$

c.  $\mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\}$

d.  $|\tilde{X}[k]| = |\tilde{X}[-k]|$

e.  $ang\ \tilde{X}[k] = -ang\ \tilde{X}[-k]$

f.  $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n]) \overset{DFS}{\longleftrightarrow} \mathcal{R}e\{\tilde{X}[k]\}$

g.  $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[n]) \overset{DFS}{\longleftrightarrow} j\mathcal{I}m\{\tilde{X}[k]\}$

where $x_e[n]$ is an even sequence, $x_o[n]$ is an odd sequence, $\mathcal{R}e\{\ \}$ denotes the real part and $\mathcal{I}m\{\ \}$ denotes the imaginary part.

(v) PERIODIC CONVOLUTION

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences each with period $N$ and with discrete Fourier series coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively. If we form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k] \tag{6.55}$$

then the periodic sequence $\tilde{x}_3[n]$ with Fourier series coefficients is

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \tag{6.56}$$

**Proof**:

Consider the sequence $\tilde{x}_3[n]$ defined as in 6.56. The DFS coefficients $\tilde{X}_3[k]$ is

$$\tilde{X}_3[k] \triangleq \sum_{n=0}^{N-1} \tilde{x}_3[n]e^{-j2\pi kn/N} \tag{6.57}$$

$$= \sum_{n=0}^{N-1} \left\{ \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \right\} e^{-j2\pi kn/N} \tag{6.58}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \sum_{n=0}^{N-1} \tilde{x}_2[n-m]e^{-j2\pi kn/N} \tag{6.59}$$

From the shifting property

$$\sum_{n=0}^{N-1} \tilde{x}_2[n-m]e^{-j2\pi kn/N} = e^{-j2\pi km/M} \tilde{X}_2[k] \tag{6.60}$$

From 6.59 and 6.60, we get

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k] \tag{6.61}$$

Q.E.D.

In summary,

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k] \tag{6.62}$$

The left-hand side of the above equation is known as periodic convolution.

Periodic Convolution

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ with the same period $N$. The periodic convolution

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \tag{6.63}$$

is calculated as follows:

1.    Form the inverted sequence $\tilde{x}_2[n-m] = \tilde{x}_2[-(m-n)]$ and the product $\tilde{x}_1[m]\tilde{x}_2[-(m-n)]$ for various values of $n$.

2.    The sum is carried over the finite interval $0 \leq m \leq N-1$.

**Note:**

The values of $\tilde{x}_2[n-m]$ in the interval $0 \leq m \leq N-1$ repeat periodically for $m$ outside that interval.
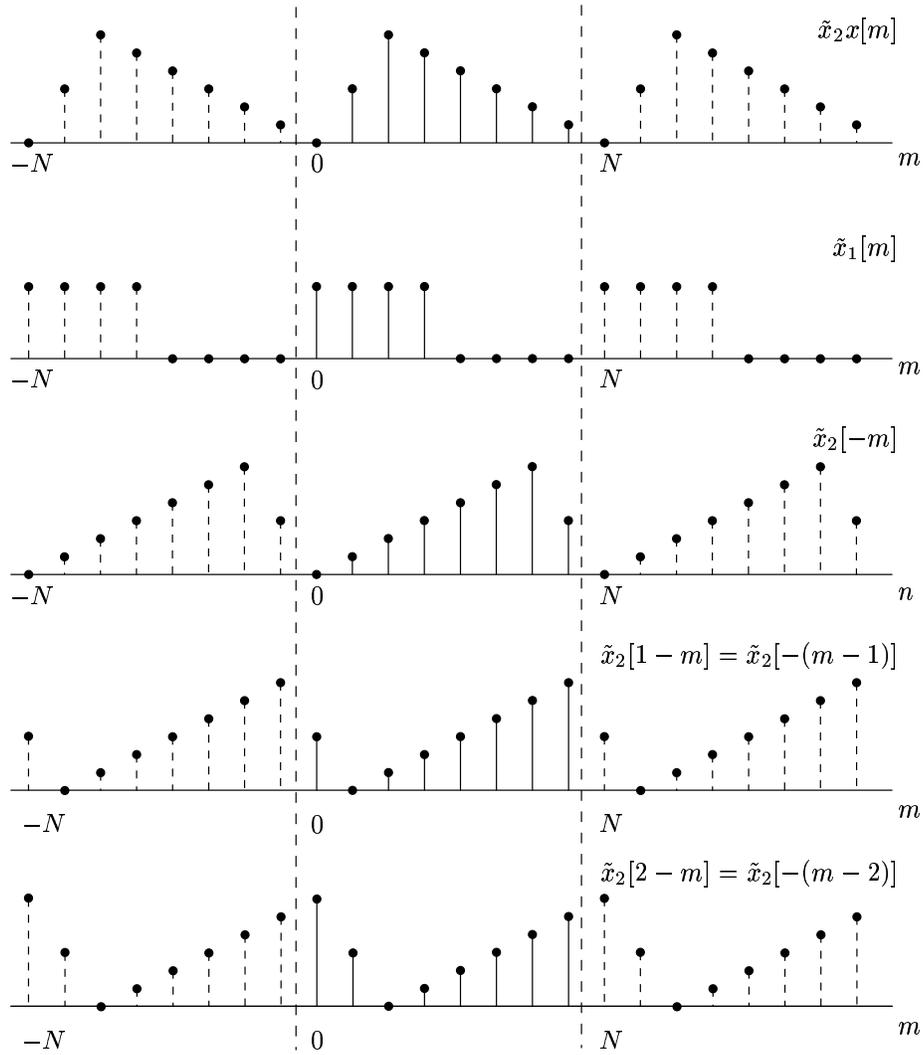
**Example 5**: Periodic Convolution



Figure 6.6: Procedure for forming the periodic convolution of two periodic sequences.

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**Corollary**:

From the duality theorem, if we exchange the roles of time and frequency, we obtain an identical result to the periodic convolution, that is,

$$\text{If } \tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n] \tag{6.64}$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are periodic sequences each with period $N$, then the DFS coefficients are given by

$$\tilde{X}_3[k] = \frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}_1[l]\tilde{X}_2[k-l] \tag{6.65}$$

The proof of the above result is similar to the proof of periodic convolution in the time domain we provided before. The reader should justify the above result. It is to be noted that the convolution

$$\sum_{l=0}^{N-1}\tilde{X}_1[l]\tilde{X}_2[k-l] \tag{6.66}$$

is also periodic with period $N$.

| Periodic Sequence (Period N) | DFS Coefficients (Period N) |
|---|---|
| 1. $\tilde{x}[n]$ | $\tilde{X}[k]$ periodic with period $N$ |
| 2. $\tilde{x}_1[n]$, $\tilde{x}_2[n]$ | $\tilde{X}_1[k]$, $\tilde{X}_2[k]$ periodic with period $N$ |
| 3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$ | $a\tilde{X}_1[k] + b\tilde{X}_2[k]$ |
| 4. $\tilde{X}[n]$ | $N\tilde{x}[-k]$ |
| 5. $\tilde{x}[n-m]$ | $W_N^{km}\tilde{X}[k]$ |
| 6. $W_N^{-ln}\tilde{x}[n]$ | $\tilde{X}[k-l]$ |
| 7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ | $\tilde{X}_1[k]\tilde{X}_2[k]$ |
| 8. $\tilde{x}_1[n]\tilde{x}_2[n]$ | $\frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}_1[l]\tilde{X}_2[k-l]$ |
| 9. $\overline{\tilde{x}}[n]$ | $\overline{\tilde{X}}[((-k))_N]$ |
| 10. $\overline{\tilde{x}}[((-n))_N]$ | $\overline{\tilde{X}}[k]$ |
| 11. $\mathcal{R}e\{\tilde{x}[n]\}$ | $\tilde{X}_{ep}[k] = \frac{1}{2}\{\tilde{X}[k] + \overline{\tilde{X}}[-k]\}$ |
| 12. $j\mathcal{I}m\{\tilde{x}[n]\}$ | $\tilde{X}_{op}[k] = \frac{1}{2}\{\tilde{X}[k] - \overline{\tilde{X}}[-k]\}$ |
| 13. $\tilde{x}_{ep}[n] = \frac{1}{2}\{\tilde{x}[n] + \overline{\tilde{x}}[-n]\}$ | $\mathcal{R}e\{\tilde{X}[k]\}$ |
| 14. $\tilde{x}_{op}[n] = \frac{1}{2}\{\tilde{x}[n] - \overline{\tilde{x}}[-n]\}$ | $j\mathcal{I}m\{\tilde{X}[k]\}$ |
| 15. Symmetry properties (for real $x[n]$) | $\begin{cases} \tilde{X}[k] = \overline{\tilde{X}}[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ |\tilde{X}[k]| = |\tilde{X}[-k]| \\ \sphericalangle\{\tilde{X}[k]\} = -\sphericalangle\{\tilde{X}[-k]\} \end{cases}$ |
| 16. $\tilde{x}_{ep}[n] = \frac{1}{2}\{\tilde{x}[n] + \tilde{x}[-n]\}$ (for real $x[n]$) | $\mathcal{R}e\{\tilde{X}[k]\}$ |
| 17. $\tilde{x}_{op}[n] = \frac{1}{2}\{\tilde{x}[n] - \tilde{x}[-n]\}$ (for real $x[n]$) | $j\mathcal{I}m\{\tilde{X}[k]\}$ |

# 6.3    The Discrete Fourier Transform (DFT)

Consider a finite-duration sequence $x[n]$ of length $N$. Form the periodic sequence $\tilde{x}[n]$ such that

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN] \qquad (6.67)$$

where $r$ is an integer.
Therefore,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \qquad (6.68)$$

Recall that

$$\tilde{x}[n] \quad \triangleq \quad \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N} \qquad (6.69)$$

$$\tilde{X}[k] \quad = \quad \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N} \qquad (6.70)$$

where both $\tilde{x}[n]$ and $\tilde{X}[k]$ are periodic with period $N$.

Now $x[n]$ is equal to one period of $\tilde{x}[n]$. To maintain a duality between the time domain and frequency domain, we choose the Fourier Coefficients that are associated with $x[n]$ to be a finite-duration sequence corresponding to one period of $\tilde{X}[k]$. This sequence will be referred to as $X[k]$, i.e.,

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases} \qquad (6.71)$$

<u>The DFT equations</u>

$$\text{Analysis: } X[k] \quad = \quad \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \; 0 \leq k \leq N-1 \qquad (6.72)$$

$$\text{Synthesis: } x[n] \quad = \quad \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, \; 0 \leq n \leq N-1 \qquad (6.73)$$

Symbolically, we will write the above equations as

$$x[n] \stackrel{DFT}{\longleftrightarrow} X[k] \qquad (6.74)$$

<u>Periodicity of the DFT</u>

Question: Is $x[n]$ defined for $n > N$?
Answer: Unfortunately, yes.

To see that $x[n]$ is still defined for $n > N$, we substitute in 6.73 for values of $n > N$, i.e.,

$$x[N+m] \;=\; \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi k(N+m)/N} \tag{6.75}$$

$$=\; \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi km/N} \cdot \underbrace{e^{j2\pi k}}_{1} \tag{6.76}$$

$$\equiv\; x[m] \tag{6.77}$$

Hence, while we consider the DFT representation valid for finite-duration sequences, we cannot ignore the inherent *periodicity* resulting from the DFT definitions. As we will show next, all of the properties of the DFT can be deduced from the DFS by restricting the observation in the time-domain $n$ and the frequency domain $k$ to a window of length $N$. Specifically, $0 \le n \le N-1$ and $0 \le k \le N-1$.

# 6.4  Properties of the DFT

(i) LINEARITY
Consider two sequences $x_1[n]$ and $x_2[n]$. Adjust the two sequences so that they will have the same length, $N$ (i.e., add zeros to the shorter sequence). Now,

$$\text{If } x_1[n] \;\overset{DFT}{\longleftrightarrow}\; X_1[k],\; 0 \le k \le N-1 \tag{6.78}$$

$$\text{and } x_2[n] \;\overset{DFT}{\longleftrightarrow}\; X_2[k] \tag{6.79}$$

$$\text{then } ax_1[n] + bx_2[n] \;\overset{DFT}{\longleftrightarrow}\; aX_1[k] + bX_2[k],\; 0 \le k \le N-1 \tag{6.80}$$

(ii) CIRCULAR SHIFT
Let

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{6.81}$$

We can write $\tilde{x}[n]$ as

$$\tilde{x}[n] = x[n \bmod N] \tag{6.82}$$

where $n \bmod N$ denotes modulo $N$ summation

**Note**:

$$
\begin{aligned}
a &= b \bmod N \; if \\
b &= a + iN, \; i = \text{integer}
\end{aligned}
$$

Define

$$
\tilde{x}_1[n] - \tilde{x}[n - m] = x[(n - m) \bmod N] \tag{6.83}
$$

Recall that

$$
\tilde{X}_1[k] = e^{-j2\pi km/N} \tilde{X}[k] \tag{6.84}
$$

$$
\begin{aligned}
\text{Now if } x[n] &= \begin{cases} \tilde{x}[n], & 0 \le n \le N - 1 \\ 0, & \text{otherwise} \end{cases} \\
\text{then } x_1[n] &= \begin{cases} \tilde{x}_1[n] = \tilde{x}[n - m] = x[(n - n) \bmod N], & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Also, from the definitions,

$$
\begin{aligned}
X[k] &= \tilde{X}[k], \; 0 \le k \le N - 1 \tag{6.85} \\
X_1[k] &= \tilde{X}_1[k], \; 0 \le k \le N - 1 \tag{6.86}
\end{aligned}
$$

Hence, the circular shift effect can be written as follows:

$$
\text{If } x[n] \overset{DFT}{\longleftrightarrow} X[k] \tag{6.87}
$$

then

$$
x[(n - m) \bmod N], \; 0 \le n \le N - 1 \overset{DFT}{\longleftrightarrow} e^{-j2\pi km/N} \tag{6.88}
$$

How to Perform Circular Shift?

   1.    Construct $\tilde{x}[n]$ from $x[n]$

   2.    $x[(n - m) \bmod N = \tilde{x}[n - m]$, so perform the shift
         on $\tilde{x}[n]$ and consider only $0 \le n \le N - 1$.

**Example 6**: Circular shift



$x[n]$

$(a)$

$\tilde{x}[n]$

$(b)$

$\tilde{x}_1[n] = \tilde{x}[n+2]$

$(c)$

$$x_1[n] = \begin{cases} \tilde{x}_1[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$
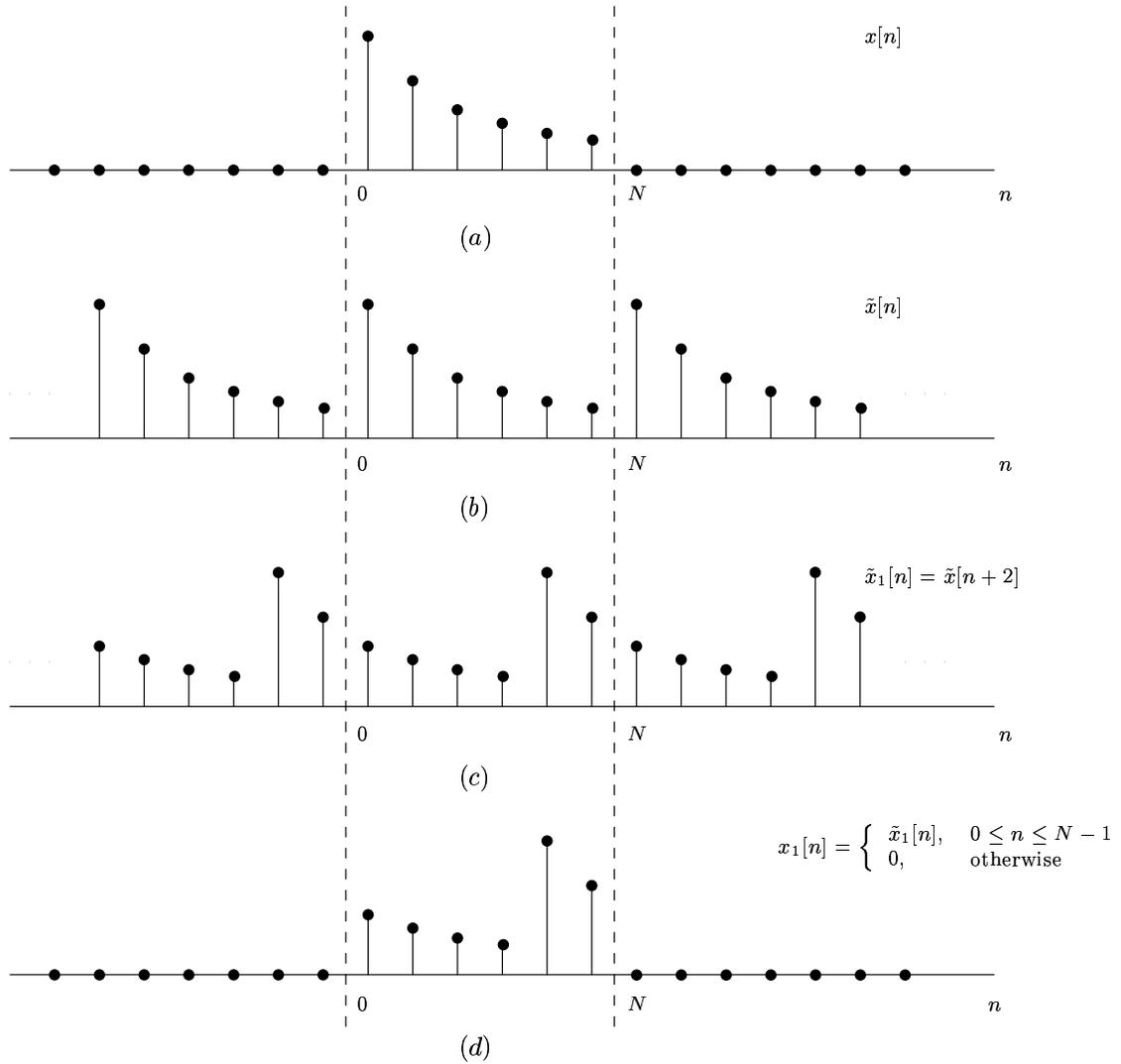
$(d)$

Figure 6.7: Circular shift of a finite-length sequence; i.e., the effect in the same time domain of multiplying the DFT of the sequence by a linear phase factor.

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(iii) DUALITY OF THE DFT COEFFICIENTS

$$\text{If } x[n] \quad \overset{DFT}{\longleftrightarrow} \quad X[k]$$

$$\text{then } X[n] \quad \overset{DFT}{\longleftrightarrow} \quad Nx[-k \text{ mod } N], \ 0 \le k \le N - 1$$

**Note**:

For $0 \le n \le N - 1$, the following relations hold

$$n \text{ mod } N \;=\; n$$
$$-n \text{ mod } N \;=\; N - n$$

(iv) CIRCULAR CONVOLUTION

Consider two finite-duration sequences $x_1[n]$ and $x_2[n]$ of length $N$. If $x_1[n] \overset{DFT}{\longleftrightarrow} X_1[k]$ and $x_2[n] \overset{DFT}{\longleftrightarrow} X_2[k]$.

Form

$$X_3[k] \;=\; X_1[k]X_2[k] \ 0 \le k \le N - 1$$

then

$$x_3[n] \;=\; \sum_{m=0}^{N-1} x_1[m]x_2[(n - m) \text{ mod } N]$$

$$\;=\; \sum_{m=0}^{N-1} x_2[m]x_1[(n - m) \text{ mod } M]$$

**Proof**:

From the DFS equation (periodic convolution)

$$\text{If } \tilde{X}_3[k] \;=\; \tilde{X}_1[k]\tilde{X}_2[k] \text{ then}$$

$$\tilde{x}_3[k] \;=\; \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n - m]$$

Now let,

$$x_1[n] \;=\; \begin{cases} \tilde{x}_1[n], & 0 \le n \le N - 1 \\ 0, & \text{otherwise} \end{cases}$$

$$x_2[n] \;=\; \begin{cases} \tilde{x}_2[n], & 0 \le n \le N - 1 \\ 0, & \text{otherwise} \end{cases}$$

i.e.,

$$\begin{aligned}
\tilde{x}_1[n] &= x_1[n \bmod N] \\
\tilde{x}_2[n] &= x_2[n \bmod N]
\end{aligned}$$

Let,

$$\begin{aligned}
x_3[n] &= \begin{cases} \tilde{x}_3[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Since,

$$\begin{aligned}
\tilde{x}_1[m] &= x_1[m] \text{ for } 0 \le m \le N-1 \\
\tilde{x}_2[n-m] &= x_2[(n-m) \bmod N] \text{ for } 0 \le m \le N-1
\end{aligned}$$

Therefore,

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[(n-m)/bmodN]$$

This relation is denoted by *circular convolution*. We will use the symbol $\circledast$ instead of $*$ to denote circular convolution, i.e.,

$$\begin{aligned}
x_3[n] &= x_1[n] \circledast x_2[n] \\
&= \sum_{m=0}^{N-1} x_1[m]x_2[(n-m) \bmod N]
\end{aligned}$$

**Example 7**:

$$\begin{aligned}
x_1[n] &= \delta[n-1] \\
x_2[n] &= \begin{cases} a^{-n}, & o \le n \le 4 \\ 0, & \text{otherwise} \end{cases} \\
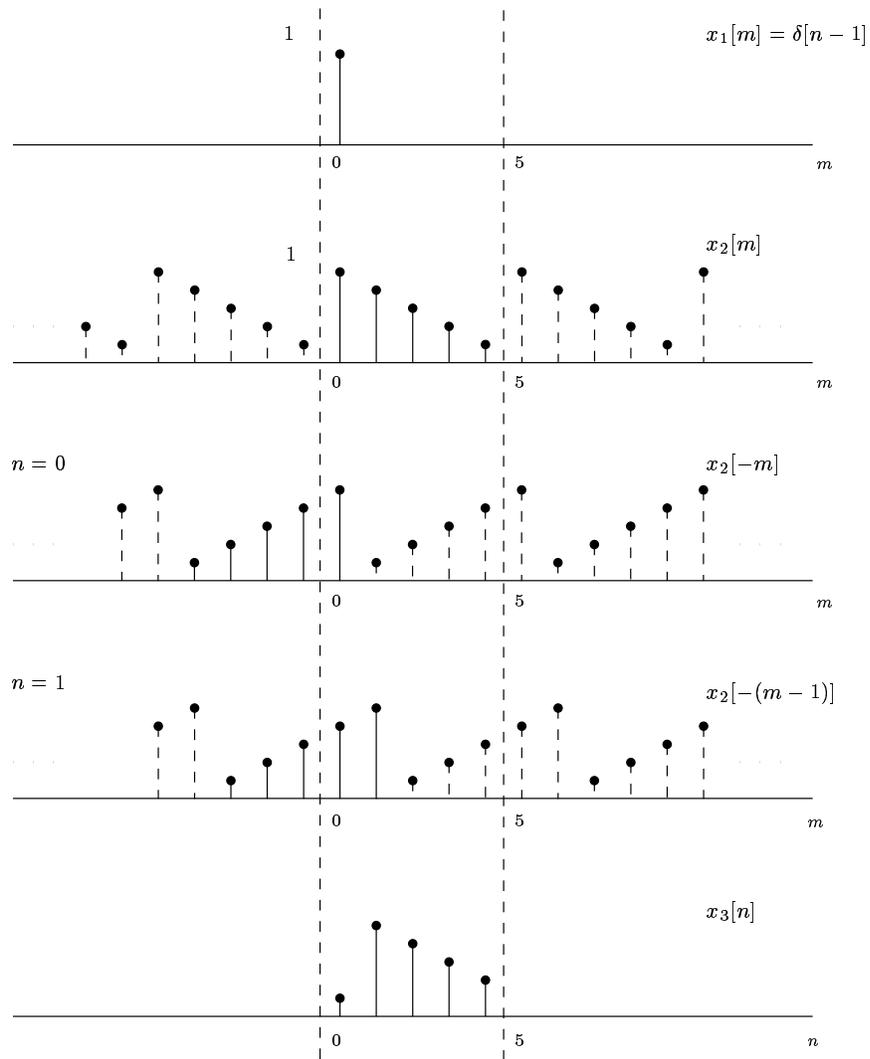N &= 5.
\end{aligned}$$

Figure 6.8: Circular Convolution for Example 7

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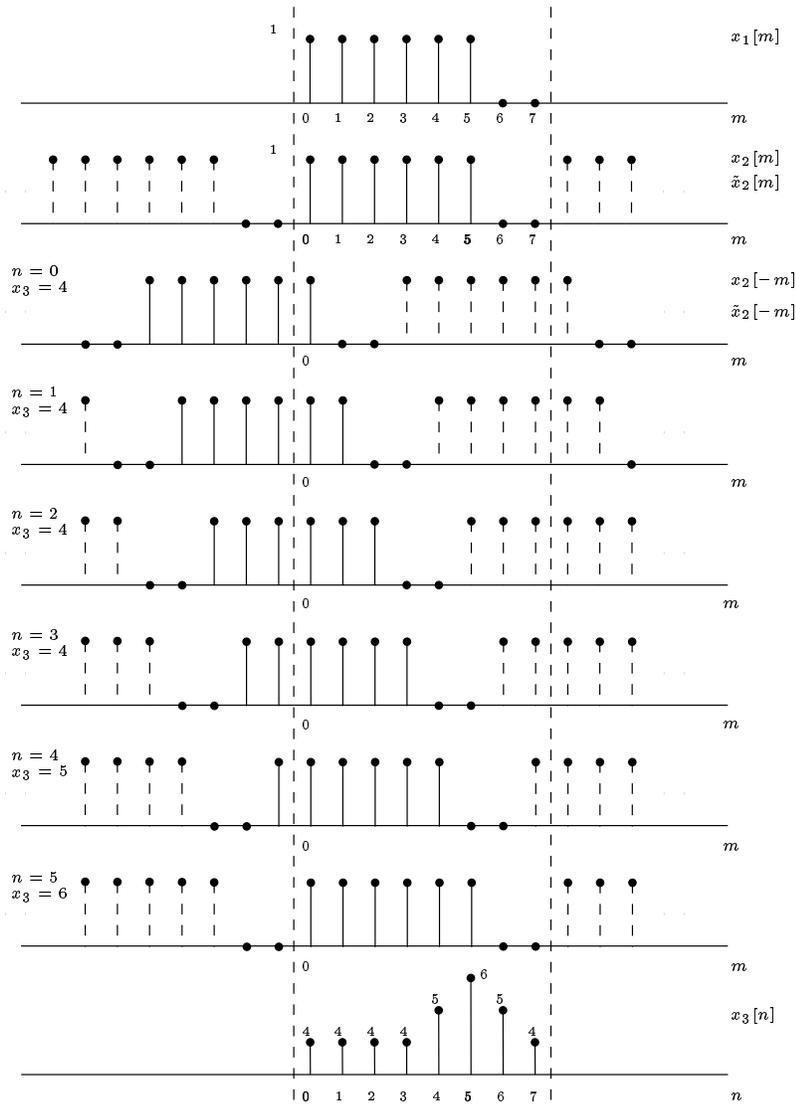**Example 8**: Consider the circular convolution below for $N = 8$.



Figure 6.9: Example 8, Circular Convolution

$\diamond \diamond \diamond$

**Example 9**: Same as example 8 with shorter sequences.
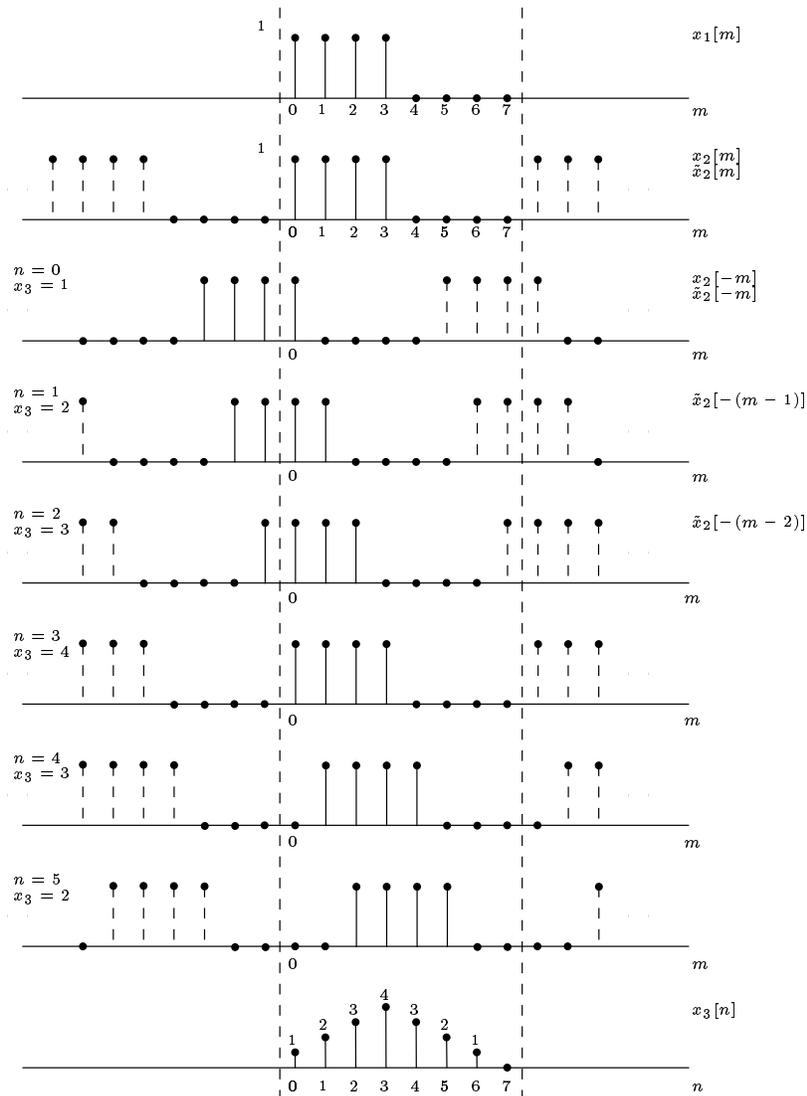


Figure 6.10: Circular Convolution, Example 9

$\Diamond \, \Diamond \, \Diamond$

# 6.5  Linear Convolution using the DFT

1. *Linear Convolution of two sequences*

Consider two sequences $x_1[n]$ and $x_2[n]$ of lengths $L$ and $P$ points, respectively. Suppose we wish to combine these two sequences by linear convolution to obtain a third sequence by linear convolution, i.e.,

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] \qquad (6.89)$$

Clearly, the product $x_1[m]x_2[n-m]$ is zero for all $m$ whatever $n < 0$ and $n > L+P-2$. Therefore, $(L+P-1)$ is the maximum length of the sequence $x_3[n]$ resulting from the Linear Convolution of a sequence of length $L$ with a sequence of length $P$.

2. *Linear Convolution using the DFT*

The DFT provides the circular convolution. To make the circular convolution similar to the linear convolution, we pad the two sequences $x_1[n]$ and $x_2[n]$ with zeros to make the two sequences have a length of $L+P-1$. The following example illustrates this idea.
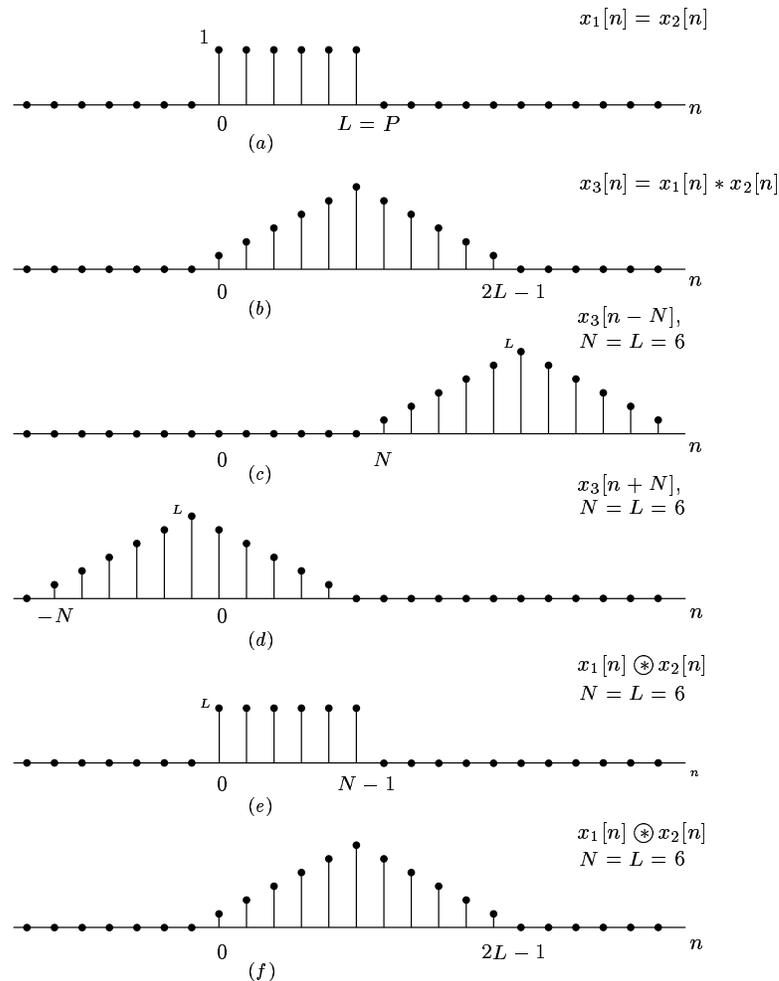
**Example 10**: Linear Convolution using the DFT



Figure 6.11: Illustration that circular convolution is equivalent to linear convolution followed by aliasing. (a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved. (b) The linear convolution of $x_1[n]$ and $x_2[n]$. (c) $x_3[n - N]$ for $N = 6$. (d) $x_3[n + N]$ for $N = 6$. (e) $x_1[n] \circledast x_2[n]$, which is equal to the sum of (b), (c), and (d) in the interval $0 \leq n \leq 5$. (f) $x_1[n] \circledast x_2[n]$.

$\diamond \diamond \diamond$

| Finite-Length Sequence (Length N) | N-Point DFT (LengthN) |
|---|---|
| 1. $x[n]$ | $X[k]$ |
| 2. $x_1[n]$, $x_2[n]$ | $X_1[k]$, $X_2[k]$ |
| 3. $ax_1[n] + bx_2[n]$ | $aX_1[k] + bX_2[k]$ |
| 4. $X[n]$ | $Nx[((-k)_N]$ |
| 5. $x[((n-m))_N]$ | $W_N^{km}X[k]$ |
| 6. $W_N^{-ln}x[n]$ | $X[((k-l))_N]$ |
| 7. $\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$ | $X_1[k]X_2[k]$ |
| 8. $x_1[n]x_2[n]$ | $\frac{1}{N}\sum_{l=0}^{N-1} X_1(l)X_2[((k-1))_N]$ |
| 9. $\overline{x}[n]$ | $\overline{X}[((-k))_N]$ |
| 10. $\overline{x}[((-n))_N]$ | $\overline{X}[k]$ |
| 11. $\mathcal{R}e\{x[n]\}$ | $X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + \overline{X}[((-k))_N]\}$ |
| 12. $j\mathcal{I}m\{x[n]\}$ | $X_{op}[k] = \frac{1}{2}\{X[((k))_N] - \overline{X}[((-k))_N]\}$ |
| 13. $x_{ep}[n] = \frac{1}{2}\{x[n] + \overline{x}[((-n))_N]\}$ | $\mathcal{R}e\{X[k]\}$ |
| 14. $x_{op}[n] = \frac{1}{2}\{x[n] - \overline{x}[((-n))_N]\}$ | $j\mathcal{I}m\{X[k]\}$ |
| 15. Symmetry properties (for real $x[n]$) | $\begin{cases} X[k] = \overline{X}[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ |X[k]| = |X[((-k))_N]| \\ \text{“}\{X[k]\} = -\text{“}\{X[((-k))_N]\} \end{cases}$ |
| 16. $x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$ (for real $x[n]$) | $\mathcal{R}e\{X[k]\}$ |
| 17. $x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$ (for real $x[n]$) | $j\mathcal{I}m\{X[k]\}$ |

Figure 6.12: Summary of Properties of the Discrete Fourier Transform