Chapter 7

On Power Spectral Estimation

In this section, we discuss a few aspects of the power spectral estimation problem. The results will be used in signal/image restoration. We will start with the one-dimensional case before the 2-D.

7.1 1-D spectral Estimation

Definition: Let $X(\omega, t)$ (henceforth will be denoted by X) be a random process that is w.s.s. Let $R_X(\tau)$ be the autocorrelation function. The power spectrum density function $S_X(f)$ is the Fourier transform of $R_X(\tau)$; i.e.

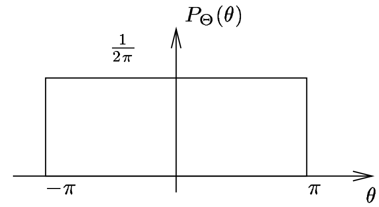
$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau. \qquad \diamondsuit$$

The above definition is also known as the Weiner-Kinchin theorem.

Example 1:

$$X(t, \omega) = A\cos(\omega_o t + \Theta), \quad \Theta \sim U(-\pi, \pi) \tag{7.1}$$

where A and ω_o are constants.



$$
\begin{aligned}
E\left[X(\omega,t)\right] &= \int_{-\pi}^{\pi} A\cos(\omega_o t+\theta)\frac{1}{2\pi}d\theta = 0 \\
R_X(t_1,t_2) &\triangleq E\left[X(\omega,t_1)X(\omega,t_2)\right] \\
&\equiv E\left[A^2\cos(\omega_o t_1+\Theta)\cos(\omega_o t_2+\Theta)\right] \\
&= \frac{A^2}{2}E\left[\cos\omega_o(t_1-t_2)+\cos(\omega_o(t_1+t_2)+2\Theta)\right] \\
&= \frac{A^2}{2}\cos\omega_o(t_1-t_2)+\frac{A^2}{2}\underbrace{E\left[\cos(\omega_o(t_1+t_2)+2\Theta)\right]}_{0} \\
&\equiv \frac{A^2}{2}\cos\omega_o(t_1-t_2) \\
&\triangleq \frac{A^2}{2}\cos\omega_o t, \qquad \tau = \mid t_1-t_2\mid \\
&= R_X(\tau) \\
S_X(f) &= \frac{A^2}{2}\int_{-\infty}^{\infty}\cos\omega_o\tau e^{-j2\pi f\tau}d\tau \\
&= \frac{A^2}{2}\left[\delta(f-\frac{\omega_o}{2\pi})+\delta(f+\frac{\omega_o}{2\pi})\right]
\end{aligned}
$$

$\diamond\diamond\diamond$

Definition: Consider a zero-mean w.s.s. discrete-parameter random process $X(\omega,t_i)=\{X_i\}$; i.e.,

$$
\begin{aligned}
E\left[X_i\right] &= 0 \\
E\left[X_iX_j\right] &\triangleq R_X(i,j) \quad \equiv \quad R_X(i-j) \\
&= R_X(j-i) \quad \triangleq \quad R_X(m).
\end{aligned}
$$

The power spectrum density $S(f_d)$ is defined as

$$S_X(f_d) = \sum_{m=-\infty}^{\infty} R_X(m) e^{-j2\pi f_d m} \qquad \diamondsuit \tag{7.2}$$

Notes:

1) $S_X(f_d)$ is periodic with period $= 1$; i.e.; $S_X(f_d + 1) = S_X(f_d)$

2) $R_X(m)$ is a discrete sequence, but $S_X(f_d)$ is continuous function of f_d.

3) If the process $\{X_i\}$ is obtained by sampling the process $X(t)$ at a rate f_s samples/sec., then $f_d = f/f_s$. That is, all that we studied about sequence Fourier transforms holds for random sequences.

7.2 Classical Methods for Power Spectral Estimation

7.2.a Periodogram

Consider N samples $\{x[n],\ n = 0, 1, 2, ..., N-1\}$ from an ergodic random process $X(\omega, t_i) = \{X_i\}$. We can estimate the *mean* and the *autocorrelation* function of $\{X_i\}$ from the samples $\{x[n]\}$ as follows:

$$\hat{\mu} \triangleq \frac{1}{N} \sum_{n=0}^{N-1} x[n]; \tag{7.3}$$

sample mean

$$\hat{R}_N[k] = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-|k|-1} x[n]x[n+|k|], & k = 0, \pm 1, \pm 2, ..., \pm (N-1) \\ 0, & |k| \geq N; \end{cases} \tag{7.4}$$

sample autocorrelation function

Notes:

1. $E[\hat{\mu}_N] = \frac{1}{N} \sum_{n=0}^{N-1} E[x[n]] = \frac{1}{N} \sum_{n=0}^{N-1} = \frac{N\mu}{N} = \mu$
Hence, 7.3 is an *unbiased* estimator of the mean.

2. If the process $X(w, t_i)$ is white, then the samples$\{X_i\}$ are iid (independent and identically distributed).

$$\begin{aligned}
var(\hat{\mu}_N) &= E\left[\hat{\mu}_N - E(\hat{\mu}_N)\right]^2 \\
&= E\left[\frac{1}{N}\sum_{n=0}^{N-1} x[n] - \mu\right]^2 \\
&= \frac{1}{N^2}E\left[\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(x[n]-\mu)(x[m]-\mu)\right] \\
&= \frac{1}{N^2}E\sum_{n=0}^{N-1}(x[n]-\mu)^2 + \frac{1}{N^2}\underbrace{E\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(x[n]-\mu)(x[m]-\mu)}_{0,\quad n\neq m} \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}E\left[x[n]-\mu\right]^2 \\
&\stackrel{\triangle}{=} \frac{1}{N^2}\sum_{n=0}^{N-1}\sigma^2 \\
&\equiv \frac{\sigma^2}{N}
\end{aligned}$$

$$\lim_{N\to\infty} var(\hat{\mu}_N) = 0 \qquad \text{The estimate in 7.3 is } \textit{consistent} \tag{7.5}$$

3.

$$\begin{aligned}
E\left[\hat{R}_N[k]\right] &= \begin{cases} \frac{1}{N}\sum_{n=0}^{N-|k|-1} E\left[x[n]+x[n+|k|]\right], & k = 0, \pm 1, \pm 2, ..., \pm(N-1) \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} \frac{1}{N}\sum_{n=0}^{N-|k|-1} R[k], & k = 0, \pm 1, \pm 2, ..., \pm(N-1) \\ 0, & \text{else} \end{cases} \\
&= \frac{N-|k|}{N}R[k] \qquad (7.6) \\
&\neq R[k] \quad \text{except for } k = 0
\end{aligned}$$

Therefore, 7.4 is *not* an unbiased estimate for $R[k]$.

From 7.6, $\lim_{N\to\infty} E\left[\hat{R}_N[k]\right] \to 0$

Therefore, 7.4 is an *asymptotically unbiased estimator for* $R[k]$.

4. It can be shown that

$$var(R_N[k]) \longrightarrow_{n\to\infty} 0 \tag{7.7}$$

Therefore, 7.4 is a *consistent estimator for* $R[k]$.
But, this is not true for every k.

5. An *unbiased* and *consistent* estimator for $R[k]$ can be defined as follows:

$$\hat{R}'_N[k] = \begin{cases} \frac{1}{N-|k|}\sum_{n=0}^{N-|k|-1} x[n]x[n+|k|], & k = 0, \pm 1, \pm 2, ..., \pm(N-1) \\ 0, & \text{else} \end{cases} \tag{7.8}$$

However, $var(R'_N[k]) > var(R_N[k])$. Hence, we will use the 7.4 instead of 7.8 to estimate $R[k]$.

7.2.b Periodogram

From the finite sequence $\{x[n], \quad n = 0, 1, ..., N-1\}$, define the infinite sequence

$$x_N[n] \triangleq \begin{cases} x[n], & n = 0, 1, 2, ..., N-1 \\ 0, & n < 0 \text{ and } n \leq N \end{cases} \tag{7.9}$$

Hence, 7.4 becomes

$$\hat{R}_N[k] = \frac{1}{N}\sum_{n=-\infty}^{\infty} x_N[n]x_N[n+k] \tag{7.10}$$

Now from 7.2,

$$S_X(f_d) = \sum_{m=-\infty}^{\infty} R[m]e^{-j2\pi f_d m} \tag{7.11}$$

Hence,

$$\begin{aligned}
\hat{S}_X(f_d) &= \sum_{k=-\infty}^{\infty} \hat{R}_N[k] e^{-j2\pi f_d k} && (7.12)\\
&= \frac{1}{N} \sum_{k=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x_N[n] x_N[n+k] \right] e^{-j2\pi f_d k}\\
&= \frac{1}{N} \left[\sum_{n=-\infty}^{\infty} x_N[n] e^{j2\pi n f_d} \right] \left[\sum_{k=-\infty}^{\infty} x_N[n+k] e^{-j2\pi f_d (k+n)} \right]\\
&\equiv \frac{1}{N} \left[\sum_{n=-\infty}^{\infty} x_N[n] e^{j2\pi f_d n} \right] \left[\sum_{m=-\infty}^{\infty} e^{-j2\pi f_d m} \right]\\
&\triangleq \frac{1}{N} \overline{X_N(f_d)} X_N(f_d),\\
\text{where} \quad X_N(f_d) &= \sum_{n=-\infty}^{\infty} x_N[n] e^{-j2\pi f_d n}
\end{aligned}$$

Therefore,

$$\hat{S}_X(f_d) = \frac{1}{N} |X_N(f_d)|^2 \tag{7.13}$$

Equation 7.13 is known as the periodogram.

Is the periodogram a good estimate for the Power Spectral Density?
NO!
To see why we should not use it, let's check consistency.
From 7.12 and using N-samples,

$$\hat{S}_X(f_d) = \sum_{k=-(N-1)}^{(N-1)} \hat{R}_N[k] e^{-j2\pi f_d k} \tag{7.14}$$

From 7.4,

$$\hat{R}_N[k] = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-|k|-1} x[n] x[n+|k|], & k = 0, \pm 1, \pm 2, ..., \pm (N-1) \\ 0, & |k| \geq N \end{cases} \tag{7.15}$$

Now for finite N,

$$\begin{aligned}
E\left[\hat{S}_X(f_d)\right] &= \sum_{k=-(N-1)}^{N-1} E\left[\hat{R}_N[k]\right] e^{-j2\pi f_d k} \\
&= \sum_{k=-(N-1)}^{N-1} \left(1-\frac{|k|}{N}\right) E\left[x[n]x[n+|k|]\right] e^{-j2\pi f_d k} \\
&= \sum_{k=-(N-1)}^{N-1} \left(1-\frac{|k|}{N}\right) R[k] e^{-j2\pi f_d k} \\
&\neq S_X(f_d) = \sum_{k=-(N-1)}^{N-1} R[k] e^{-j2\pi f_d k}
\end{aligned}$$

i.e., $\hat{S}_X(f_d)$ is NOT unbiased estimator for $S_X(f_d)$ if N is finite.

$$\text{If} \quad N \to \infty \Rightarrow \left(1-\frac{|k|}{N}\right) \to 1 \tag{7.16}$$

$$\Rightarrow E\left[\hat{S}_X(f_d)\right] = S_X(f_d), \tag{7.17}$$

i.e., $\hat{S}_X(f_d)$ in 7.14 is *asymptotically unbiased.* The trouble is consistency (i.e., how dependent is the estimate in 7.14 on the lag values of k?). In other words, is 7.4 a consistent estimator for $R[k]$ for every k? The answer is NO because of the following:

From 7.4 is $k = N-1$,

$$\hat{R}_N[k] = \frac{1}{N} x[o]x[N-1]; \tag{7.18}$$

i.e., the product of one term. If we get another record from the same process, the product in 7.18 will be different, in general. Therefore, for a fixed k (much less than N), 7.4 is a consistent estimator for $R[k]$. But, for a fixed N, we cannot claim that $R_N[k]$ in 7.4 is consistent for every $k \in [-(N-1), (N-1)]$.

Now, if $\hat{R}[k]$ in 7.4 is *not* a consistent estimator for $R[k]$, hence $\hat{S}_X(f_d)$ in 7.14 will not be a consistent estimator for $S_X(f_d)$. How can we improve the situation?

1. Averaged Periodogram
2. Smoothed Periodogram
3. Modern methods such as the maximum entropy method.

7.2.c Effect of finite data

From 7.9, the sequence $x_N[n]$ was defined as

$$x_N[n] = \begin{cases} x[n], & n = 0, 1, 2, ..., N-1 \\ 0, & \text{else} \end{cases} \tag{7.19}$$

Another representation is

$$\begin{aligned}
x_N[n] &= x[n]w[n] && (7.20) \\
\text{where } w[n] &= \begin{cases} 1, & n \in [0, N-1] \\ 0, & \text{else} \end{cases} && (7.21) \\
\text{Therefore } X_N(f_d) &= X(f_d) * W(f_d) && (7.22) \\
W(f_d) &= \sum_{n=0}^{N-1} 1 \; e^{-j2\pi f_d n} \\
&= \frac{1 - e^{-j2\pi f_d N}}{1 - e^{-j2\pi f_d}} \\
&= \frac{e^{-j2\pi f_d N/2}\left(e^{j2\pi f_d N/2} - e^{-j2\pi f_d N/2}\right)}{e^{-j2\pi f_d/2}\left(e^{j2\pi f_d/2} - e^{-j2\pi f_d/2}\right)} \\
&= e^{-j2\pi f_d (N-1)/2} \frac{\sin(\pi f_d N)}{\sin(\pi f_d)} \\
&= e^{-j\pi f_d (N-1)} \frac{\sin(\pi f_d N)}{\sin(\pi f_d)} \\
&\triangleq e^{-j\pi f_d (N-1)} R(f_d) \\
R(f_d) &\triangleq \frac{\sin(\pi f_d N)}{\sin(\pi f_d)}
\end{aligned}$$

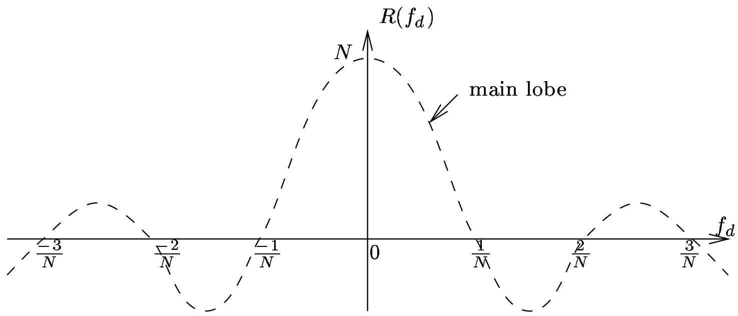
Let's plot $R(f_d)$.

From the plot of $R(f_d)$, it is clear that the width of the main lobe is inversely proportional to the number of samples N. Ideally, N is ∞ which means that $R(f_d) = S(f_d)$; i.e., 7.22 becomes

$$\begin{aligned}
\lim_{N \to \infty} X_N(f_d) &= X(f_d) * \delta(f_d) && (7.23) \\
&\equiv X(f_d); \quad \text{No distortion} && (7.24)
\end{aligned}$$

If N is finite, $X_N(f_d) = X(f_d) * W(f_d)$, i.e., a smoothed version of $X(f_d)$. The smoothing introduced by $W(f_d)$ can severely limit the *resolution*. For example, if $X(f_d)$ is known to

f_d	$R(f_d)$	
0	N	$\longleftarrow$ note : $\lim_{f_d \to 0} \frac{\sin \pi f_d N}{\sin \pi f_d} = N$
$\pm \frac{m}{N}$	0	$, m$ *an integer*



contain two spikes (sharp peaks) at f_{d_1} and f_{d_2}, we must select the number of samples N such that the width of the main lobe is smaller than $|f_{d_1} - f_{d_2}|$ in order to have the two spikes preserved in $X_N(f_d)$.

Note:

1. $w_S[n]$ is symmetric in the above figure. Therefore $W_S(f_d) = R(f_d)$ i.e., no phase shift.

2. Same conclusion if we used an asymmetric window, i.e., $w[n]$ in 7.21 instead of $w_s[n]$ as shown on the figure.

7.2.d So what is the effect of finite data?

1. Leakage or smearing or loss of resolution if the width of the main lobe (i.e., $\frac{1}{N}$) is larger than the required resolution Δf_d.

2. Side lobes introduce ripples in $X_N(f_d)$, i.e., $X_N(f_d)$ will have suprions details. That is, *distortion*.

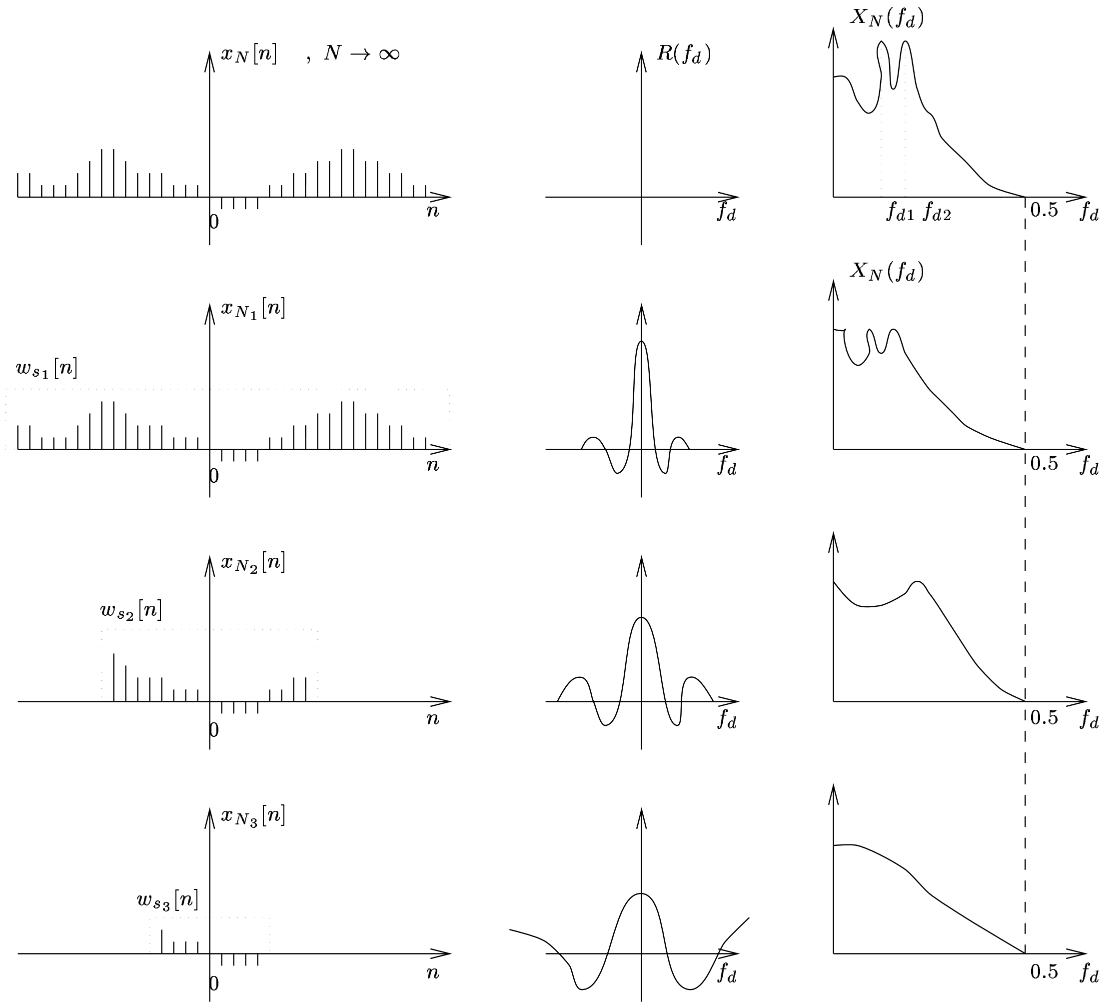


Figure 7.1: Illustration of the degradation in resolution of $X_N(f_d)$ as the number of samples N decreases.

7.2.e How to improve $X_N(f_d)$?

From 7.20

$$x_N[n] = x[n]w[n] \tag{7.25}$$

instead of $w[n]$ in 7.21 (i.e., a box), several windows have been introduced in the sixties and seventies to improve the resolution and reduce the ripple effect. Unfortunately, the two issues (resolution and ripple effects) are opposing each other. Improving one is at the expense of the other. Kaiser's windows produce an empirical (yet optimum) compromise.

7.2.f Common Windows

Definition	*SequenceFourierTransform*
Rectangular	
$W[k] = \begin{cases} 1, & k \le N-1 \\ 0, & \text{else} \end{cases}$	$W_R(f_d) = e^{-j2\pi f_d \frac{N-1}{2}} \frac{\sin(\pi f_d N)}{\sin(\pi f_d)}$
Bartlett	
$W[k] = \begin{cases} 1 - \frac{k}{N-1}, & k \le N-1 \\ 0, & \text{else} \end{cases}$	$W_B f_d = \frac{2}{N-1} e^{-j2\pi f_d \frac{N-1}{2}} \left[\frac{\sin(\pi f_d \frac{N-1}{2})}{\sin(\pi f_d)} \right]^2$
Hanning	
$W[k] = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi k}{N-1}), & k \le N-1 \\ 0, & \text{else} \end{cases}$	$W(f_d) = \left[\frac{1}{2} W_R(f_d) + \frac{1}{4} W_R(f_d - \frac{1}{N-1}) + \frac{1}{4} W_R(f_d + \frac{1}{N-1}) \right] e^{-j2\pi f_d \frac{N-1}{2}}$
Hamming	
$W[k] = \begin{cases} 0.54 + 0.46\cos(\frac{\pi k}{N-1}), & k \le N-1 \\ 0, & \text{else} \end{cases}$	$W(f_d) = e^{-j2\pi f_d \frac{N-1}{2}} \left[0.54 W_R(f_d) + 0.23 W_R(f_d - \frac{1}{N-1}) + +0.23 W_R(f_d + \frac{1}{N-1}) \right]$
Parzen	
$W[k] = \begin{cases} 2(1 - \frac{k}{N-1})^3 - (1 - 2\frac{k}{N-1})^3, & k \le \frac{N-1}{2} \\ 2(1 - \frac{k}{N-1})^3, & \frac{N-1}{2} \le N-1 \\ 0, & \text{else} \end{cases}$	$W(f_d) = e^{-j2\pi f_d \frac{N-1}{2}} \frac{64}{(N-1)^3} \left[\frac{3}{2} \frac{\sin^4 \pi f_d \frac{N-1}{4}}{\sin^4 \pi f_d} - \frac{\sin^4 \pi f_d \frac{N-1}{4}}{\sin^2 \pi f_d} \right]$

7.3 The Averaged Periodogram

7.3.a The Periodogram, Recap.

Given N samples $\{x[n], n = 0, 1, 2, ..., N-1\}$,

1. The sample mean is estimated by

$$\hat{\mu}_N = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \tag{7.26}$$

2. The sample autocorrelation is estimated by

$$R_N[k] = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-|k|-1} x[n]x[n+|k|], & k = 0, \pm 1, \pm 2, ..., \pm(N-1) \\ 0, \text{else} \end{cases} \tag{7.27}$$

3. The periodogram Spectrum Estimate

$$\hat{S}_X(f_d) = \frac{1}{N}|X_N(f_d)|^2 \tag{7.28}$$

$$\text{where} W_N(f_d) \equiv \sum_{n=0}^{N-1} x[n]e^{-j2\pi f_d n} \tag{7.29}$$

4. As we pointed out, the estimator in 7.28 is not consistent; that is, if $\hat{S}_X(f_d)$ is computed form different sets of N samples $\{x[n], n = 0, 1, 2, ..., N-1\}$, the results will be different. Therefore, if we compute only one $\hat{S}_X(f_d)$ no matter how large N is, there is no guarantee that the value of $\hat{S}_X(f_d)$ will be close to the real power spectral density $S(f_d)$. Hence, $\hat{S}_X(f_d)$ is not a consistent estimator and should not be used.

7.3.b Average Periodogram

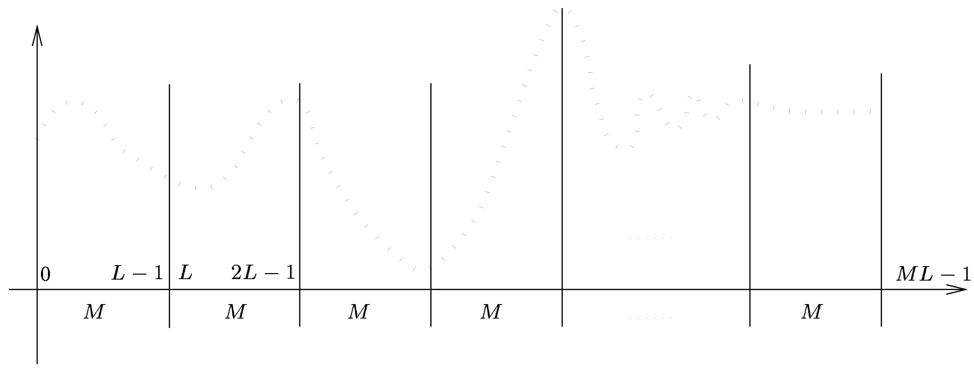
We can modify the periodogram to yield a consistent estimator of $S(f_d)$.

Consider the N-samples $\{x[n], n \in [0, N-1]\}$ is divided into L-sections, each of length $M = \frac{N}{L}$ as follows:

$$x^{(i)}(m) = x(iM + m), \quad i \in [0, L-1]$$
$$m \in [0, M-1]$$

We then compute the periodogram for each segment of length M, i.e.,

$$\hat{S}_M^{(i)}(f_d) \triangleq \frac{1}{M}\left|\sum_{m=0}^{M-1} x^{(i)}[m]e^{-j2\pi f_d m}\right|^2, \qquad i \in [0, L-1]. \tag{7.30}$$



We can show that the average of these $S_M^{(i)}(f_d)$,

$$\hat{S}_M^L(f_d) \triangleq \frac{1}{L}\sum_{i=0}^{L-i} S_M^{(i)}(f_d) \tag{7.31}$$

is a consistent estimator of the power spectral density $S(f_d)$.

Again, consistent will imply that different data records of the same random process will produce similar power spectral density estimates.

7.3.c Now what is the price of consistency?

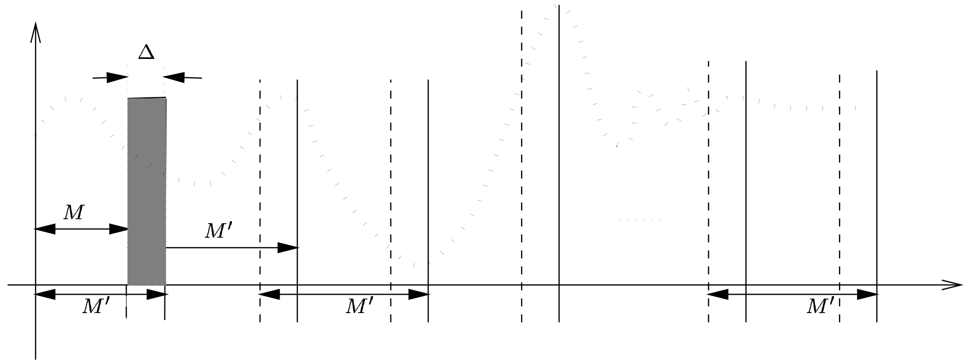
Well, we studied the effect of finite data and concluded that finite data (i.e., M is small in our case) will make the width of the main lobe in $W(f_d)$ large which will affect the resolution of $X_M(f_d)$ in 7.30.

Also, the bias in the estimate 7.31 will be larger than that in 7.28. So, the average periodogram provides for consistency at the expense of resolution and bias in the resulting estimate.

7.3.d Again, how to improve the situation?

Well, how about dividing $\{x[n], n \in [0, N-1]\}$ into L *overlapping sections*, each of length $M' > M$?

Illustration



Given N samples, select the number of sections L, instead of using sections of length $M = \frac{N}{L}$. We use sections of length $M^{'} > M$.

$$M^{'} = M + \Delta \tag{7.32}$$

where Δ is the size of the overlap region between two consecutive sections.

Since $M^{'} > M$, size of the main lobe decreases and resolution is improved. Also bias will decrease. However, the variance of the estimate in 7.31 (with M^{i} replaces M) will increase, i.e., inconsistency might be an issue again. So, there will be compromises always and that is where experience makes the difference!

7.4 The Smoothed Periodogram

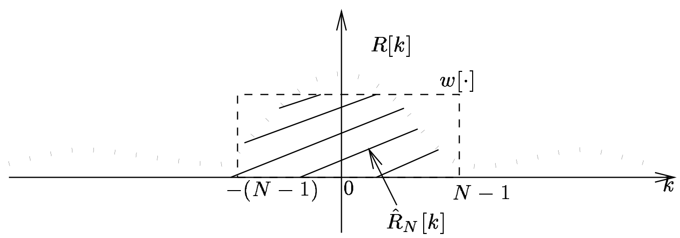
From 7.14

$$\hat{S}_X(f_d) = \sum_{k=-(N-1)}^{N-1} \hat{R}_N[k] e^{-j2\pi f_d k} \tag{7.33}$$

$$\equiv \sum_{k=-\infty}^{\infty} R[k] w[k] e^{-j2\pi f_d k} \tag{7.34}$$

That is, 7.33 is obtained by convolving $S_X(f_d)$ and $W(f_d)$.

$$\hat{S}_X(f_d) = S_X(f_d) * W(f_d) \tag{7.35}$$



We know the effect of convolution. It will smooth the spectrum and reduce the resolution. In 7.34, the window is rectangular. Other windows have been introduced to improve on the problems of rectangular windows as we studied before.

Computation procedure for FFT convolution of finite-length

1. Let $x(t)$ and $h(t)$ be finite-length functions shifted from the origin by a and b, respectively.

2. Shift $x(t)$ and $h(t)$ to the origin and sample

$$\begin{aligned} x(k) &= x(kT+a) \quad k = 0,1,...,P-1 \\ h(k) &= h(kT+b) \quad k = 0,1,...,Q-1 \end{aligned} \tag{7.36}$$

3. Choose N to satisfy the relationships

$$\begin{aligned} N &\geq P+Q-1 \\ N &= 2^{\gamma} \quad \gamma \text{ integer valued.} \end{aligned} \tag{7.37}$$

where P is the number of samples defining $x(t)$ and Q is the number of samples defining $h(t)$.

4. Augment with zeros the sampled functions of step (2).

$$\begin{aligned} x(k) &= 0 \quad k = P, P+1,...,N-1 \\ h(k) &= 0 \quad k = Q, Q+1,...,N-1 \end{aligned} \tag{7.38}$$

5. Compute the discrete transform of $x(k)$ and $h(k)$

Definition	*Sequence Fourier Transform*
Rectangular	
$W[k] = \begin{cases} 1, & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(f_d) = W_R(f_d) = \frac{\sin(\pi f_d(2M+1))}{\sin(\pi f_d)}$
Bartlett	
$W[k] = \begin{cases} 1 - \frac{\lvert k\rvert}{M}, & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(f_d) = W_B f_d = \frac{1}{M}\left[\frac{\sin(\pi f_d M)}{\sin(\pi f_d)}\right]^2$
Hanning	
$W[k] = \begin{cases} \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi k}{M}), & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(f_d) = \frac{1}{4}W_R\left(f_d - \frac{1}{M}\right) + \frac{1}{2}W_R(f_d) + \frac{1}{4}W_R\left(f_d + \frac{1}{M}\right)$
Hamming	
$W[k] = \begin{cases} 0.54 + 0.46\cos(\frac{\pi k}{M}), & \lvert k\rvert \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(f_d) = 0.23W_R\left(f_d - \frac{1}{2M}\right) + 0.54W_R(f_d) + 0.23W_R\left(f_d + \frac{1}{2M}\right)$
Parzen	
$W[k] = \begin{cases} 2(1 - \frac{\lvert k\rvert}{M})^3 - (1 - 2\frac{\lvert k\rvert}{M})^3, & \lvert k\rvert \le \frac{M}{2} \\ 2(1 - \frac{\lvert k\rvert}{M})^3, & \frac{M}{2} \le M \\ 0, & \lvert k\rvert > M \end{cases}$	$W(f_d) = \frac{8}{M^3}\left(\frac{3}{2}\frac{\sin^4 \pi f_d M/2}{\sin^4 \pi f_d} - \frac{\sin^4 \pi f_d M/2}{\sin^2 \pi f_d}\right)$

$$X(n) = \sum_{k=0}^{N-1} x(k)e^{-j2\pi nk/N} \tag{7.39}$$

$$H(n) = \sum_{k=0}^{N-1} h(k)e^{-j2\pi nk/N} \tag{7.40}$$

6. Compute the product $Y(n) = X(n)H(n)$.

7. Compute the inverse discrete transform using the forward transform

$$y(k) \sum_{n=0}^{N-1} \left[\frac{1}{N} Y(n) \right] e^{+j2\pi nk/N} \tag{7.41}$$

Computation procedure for FFT correlation of finite-length function.

1. Let $x(t)$ and $h(t)$ be finite-length functions shifted from the origin by a and b, respectively.

2. Let P be the number of samples defining $x(t)$ and Q be the number of samples defining $h(t)$.

3. Choose N to satisfy the relationships

$$\begin{aligned} N &\geq P + Q - 1 \\ N &= 2^{\gamma} \qquad \gamma \text{ integer valued.} \end{aligned} \tag{7.42}$$

4. Define $x(k)$ and $h(k)$ as follows:

$$\begin{aligned} x(k) &= 0 & k &= 0, 1, ..., N - P \\ x(k) &= x(kT + a) & k &= N - P + 1, N - P + 2, ..., N - 1 \\ h(k) &= h(kT + b) & k &= 0, 1, ..., Q - 1 \\ h(k) &= 0 & k &= Q, Q + 1, ..., N - 1 \end{aligned} \tag{7.43}$$

5. Compute the discrete transform of $x(k)$ and $h(k)$

$$X(n) = \sum_{k=0}^{N-1} x(k) e^{-j2\pi nk/N}$$
$$H(n) = \sum_{k=0}^{N} h(k) e^{-j2\pi nk/N}$$

6. Change the sign of the imaginary part of $H(n)$ to obtain $H^*(n)$.

7. Compute the product

$$Z(n) = X(n) H^*(n) \tag{7.44}$$

8. Compute the inverse transform using the forward transform

$$z(k) = \sum_{n=0}^{N-1} \left(\frac{1}{N} Z(n) \right) e^{+j2\pi nk/N} \tag{7.45}$$