ECE 530
Spring 08
HW # 8

1. $\{X_i\}$, $i=1,2,\cdots,n$ are iid Rayleigh;
i.e $f_X(x) = \frac{x}{\sigma^2} \exp(-x^2/2\sigma^2)$; $x \geq 0$.
Let $S = \max_i \{X_i\}$, $i=1,2,\cdots,n$
$\therefore$ $E(S^2) = 2\sigma^2 \sum_{k=1}^{n} \frac{1}{k}$.

Proof
First, let's get the density function of the RV S.

$$F_S(s) \triangleq P(S \leq s)$$
$$\equiv P(X_1 \leq s, X_2 \leq s, \cdots, X_n \leq s)$$
$$= P(X_1 \leq s) \cdot P(X_2 \leq s) \cdots P(X_n \leq s)$$

$\therefore$ $\boxed{F_S(s) \triangleq \left(F_X(s)\right)^n}$; X_i are iid — (1)

Differentiating w.r.t s, we get

$$\boxed{f_S(s) = n \left(F_X(s)\right)^{n-1} f_X(s)} \quad \text{— (2)}$$

Now, $F_X(x) \triangleq P(X \le x)$

$$= \int_{-\infty}^{x} f_X(\alpha)\,d\alpha \equiv \int_{0}^{x} f_X(\alpha)\,d\alpha \;;\; x \geq 0$$

$$= \int_{0}^{x} \frac{\alpha}{\sigma^2} e^{-\alpha^2/2\sigma^2}\,d\alpha$$

$$\therefore \boxed{F_X(x) = 1 - e^{-x^2/2\sigma^2}} \quad \text{—③}$$

$$\therefore F_X(s) = 1 - e^{-s^2/2\sigma^2} \quad \text{—④}$$

Hence,

$$\boxed{f_S(s) = n\left[1 - e^{-s^2/2\sigma^2}\right]^{n-1} \frac{s}{\sigma^2} e^{-s^2/2\sigma^2} \;;\; s \geq 0} \quad \text{—⑤}$$

Now,

$$E(S^2) \triangleq \int_{-\infty}^{\infty} s^2 f_S(s)\,ds$$

$$\equiv \int_{0}^{\infty} \frac{n s^3}{\sigma^2} \left[1 - e^{-s^2/2\sigma^2}\right]^{n-1} e^{-s^2/2\sigma^2}\,ds \quad \text{—⑥}$$

To simplify the integral, we note that

$$\left(1 - e^{-s^2/2\sigma^2}\right)^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i e^{-is^2/2\sigma^2} \quad \text{—⑦}$$

Therefore,

$$E(S^2) = \frac{n}{\sigma^2} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \int_0^\infty s^3 e^{-s^2/2\sigma^2(i+1)}\, ds$$

$$= \frac{n}{\sigma^2} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \frac{2\sigma^4}{(i+1)^2}$$

$$= 2\sigma^2 n \sum_{J=1}^{n} \binom{n-1}{J-1} (-1)^{J-1} \frac{1}{J^2}$$

$$= 2\sigma^2 \sum_{J=1}^{n} \frac{(-1)^{J+1}}{J} \binom{n}{J}$$

$$= 2\sigma^2 \sum_{k=1}^{n} \frac{1}{k}$$

From Tables of integrals, series and products; I.S. Gradshteyn, 1980, Academic Press, eqn. 0.155.4.

2. $X_n \xrightarrow{P} X$ iff $\lim\limits_{n\to\infty} E\left(\frac{|X_n - X|}{1+|X_n-X|}\right) = 0$

Proof:

Let $X_n - X \triangleq Y_n$. ——①

$$E\left(\frac{|Y_n|}{1+|Y_n|}\right) \triangleq \int_{-\infty}^{\infty} \frac{|Y_n|}{1+|Y_n|} f_{Y_n}(y_n)\,dy_n \quad ——②$$

$$\equiv \int_{|Y_n|\geq\varepsilon} \frac{|Y_n|}{1+|Y_n|} f_{Y_n}(y_n)\,dy_n + \int_{|Y_n|<\varepsilon} \frac{|Y_n|}{1+|Y_n|} f_{Y_n}(y_n)\,dy_n \quad ——③$$

$$\therefore E\left(\frac{|Y_n|}{1+|Y_n|}\right) \geq \int_{|Y_n|\geq\varepsilon} \frac{|Y_n|}{1+|Y_n|} f_{Y_n}(y_n)\,dy_n \quad ——④$$

$$\geq \frac{\varepsilon}{1+\varepsilon} \underbrace{\int_{|Y_n|\geq\varepsilon} f_{Y_n}(y_n)\,dy_n}_{P(|Y_n|\geq\varepsilon)} \quad ——⑤$$

$$\therefore \boxed{E\left(\frac{|Y_n|}{1+|Y_n|}\right) \geq \frac{\varepsilon}{1+\varepsilon} P(|Y_n|\geq\varepsilon)} \quad ——⑥$$

Also, from (3)

$$E\left(\frac{|Y_n|}{1+|Y_n|}\right) \le \frac{\varepsilon}{1+\varepsilon}\int_{|Y_n|\ge\varepsilon} f_{Y_n}(y_n)\,dy_n + \frac{\varepsilon}{1+\varepsilon}\int_{|Y_n|<\varepsilon} f_{Y_n}(y_n)\,dy_n. \quad (7)$$

(Check by choosing some values for ε.)

$$\le P(|Y_n|\ge\varepsilon) + \frac{\varepsilon}{1+\varepsilon}P(|Y_n|<\varepsilon); \quad (8)$$

for all $\varepsilon>0$

$$\therefore \boxed{\begin{aligned} E\left(\frac{|Y_n|}{1+|Y_n|}\right) &\le P(|Y_n|\ge\varepsilon) + \frac{\varepsilon}{1+\varepsilon}\underbrace{P(|Y_n|<\varepsilon)}_{\le 1} \quad (9)\\ &\le P(|Y_n|\ge\varepsilon) + \frac{\varepsilon}{1+\varepsilon} \quad (10)\end{aligned}}$$

Equations 6 and 10 are the lower and upper bounds of the quantity in question.

- Now let $\lim_{n\to\infty} E\left(\frac{|Y_n|}{1+|Y_n|}\right) = 0$

(i.e., assume that $X_n \xrightarrow{m.s.} X$.)

Hence, the RHS of (6) $\to 0$ as $n\to\infty$; i.e.,

$$\lim_{n\to\infty} P(|Y_n|\ge\varepsilon) \triangleq P(|X_n - X|\ge\varepsilon) = 0$$

$$\Rightarrow X_n \xrightarrow{P} X.$$

• Now let $\lim_{n\to\infty} P(|Y_n| \geq \varepsilon) = 0$

(i.e., assume that $X_n \xrightarrow{P} X$.)

From 10 by taking the limits of the two sides,

$$\therefore \lim_{n\to\infty} E\left(\frac{|Y_n|}{1+|Y_n|}\right) \leq \frac{\varepsilon}{1+\varepsilon} \quad \text{— (11)}$$

But ε is an arbitrary quantity which we can select. Therefore, we can select ε to be very small number such that

$$\lim_{n\to\infty} E\left(\frac{|Y_n|}{1+|Y_n|}\right) \longrightarrow 0.$$

$$\Rightarrow \lim_{n\to\infty} E(|Y_n|) \rightarrow 0$$

i.e., $\lim_{n\to\infty} E(|X_n - X|) \rightarrow 0$

$$\Rightarrow X_n \xrightarrow{m.s} X.$$

Q.E.D

$\boxed{E}$

3. Given $\bar{z}_x = \bar{z}_y = \bar{z}_z = 0$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1.$$

$$r_{xz} \geq r_{xy} r_{yz} - \sqrt{1-r_{xy}^2} \cdot \sqrt{1-r_{yz}^2}$$

Proof

From notes, pp. VI-13, we stated that the correlation matrix R is nonnegative definite, i.e.

$$|R| = \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix} \geq 0 , \quad \text{—①}$$

where $R_{ij} \triangleq E(X_i \overline{X_j})$

$\equiv C_{ij}$ since $\bar{z}_i = 0$ in our case

$\triangleq r_{ij} \sigma_i \sigma_j$ —②

$$\therefore R = \begin{bmatrix} r_{xx}\sigma_x^2 & r_{xy}\sigma_x\sigma_y & r_{xz}\sigma_x\sigma_z \\ r_{yx}\sigma_x\sigma_y & \sigma_{yy}^2 r_{yy} & r_{yz}\sigma_y\sigma_z \\ r_{zx}\sigma_x\sigma_z & r_{zy}\sigma_z\sigma_y & r_{zz}\sigma_{zz}^2 \end{bmatrix}$$

—③

$$\therefore R = \begin{bmatrix} 1 & r_{xy} & r_{xz} \\ r_{xy} & 1 & r_{yz} \\ r_{xz} & r_{yz} & 1 \end{bmatrix} \quad \text{— (4)}$$

note: $r_{xx} = r_{yy} = r_{zz} \equiv 1$

and, it's given that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$.

Now since the determinant of R is ≥ 0

Hence, $(1 - r_{yz}^2) - r_{xy}(r_{xy} - r_{yz} r_{xz})$

$\qquad + r_{xz}(r_{xy} r_{yz} - r_{xz}) \geq 0$

$$\therefore 1 - r_{yz}^2 - r_{xy}^2 + 2 r_{xy} r_{yz} r_{xz} \geq r_{xz}^2$$

Adding $r_{xy}^2 r_{yz}^2$ to both sides — (5)

of (5) and complete the squares on the LHS, we get

$$(1 - r_{xy}^2)(1 - r_{yz}^2) \geq (r_{xy} r_{yz} - r_{xz})^2$$

Hence, $r_{xy} r_{yz} - r_{xz} \leq \sqrt{1 - r_{xy}^2} \cdot \sqrt{1 - r_{yz}^2}$

$$\Rightarrow \boxed{r_{xz} \geq r_{xy} r_{yz} - \sqrt{1 - r_{xy}^2} \cdot \sqrt{1 - r_{yz}^2}}$$

4. Let the distance in pace i be d_i. Hence, the total distance in 100 paces will be

$$d = \sum_{i=1}^{100} d_i , \qquad \text{———} (1)$$

where d_i is iid with mean 0.97 and variance 0.01. Therefore,

$$\mu_d \equiv 100\,\mu = 97$$
$$\sigma_d^2 = 100\,\sigma^2 = 1$$

Recall: If $Y = \sum_{i=1}^{N} X_i$ and X_i are iid with mean μ_x and variance σ_x^2,

then
$$\mu_Y = 100\,\mu_x$$
$$\sigma_Y^2 = 100\,\sigma_x^2 . \quad \text{(show!)}$$

Now, we need to calculate
$$P(95 < d < 105).$$

$$P(95 < d < 105) \equiv P\left(\frac{95-\mu_d}{\sigma_d} < \frac{d-\mu_d}{\sigma_d} < \frac{105-\mu_d}{\sigma_d}\right)$$

$$= P(-2 < d_n < 8), \quad d_n \triangleq \frac{d - \bar{z}_d}{\sigma_d}$$

Now, using the Central limit theorem approximations, we can approximate the above probability as a Gaussian distribution; that is,

$$d_n \triangleq \frac{d - \bar{z}_d}{\sigma_d} \sim N(0, 1)$$

Hence,
$$\begin{aligned}
P(-2 < d_n < 8) &= G(8) - G(-2) \\
&\equiv G(8) - (1 - G(2)) \\
&= G(8) + G(2) - 1 \\
&\cong G(2) \\
&= 0.9772
\end{aligned}$$

Note: From the given table

$G(x) \cong 1$ for $x \geq 4$

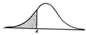
$G(y) \simeq 0$ for $y \leq -4$.

So, we took $G(8) \simeq 1$.

$\boxed{\Sigma}$

Table I Values of the standard normal distribution function

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du = P(Z \leq z)$$



z	0	1	2	3	4	5	6	7	8	9
−3.	.0013	.0010	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000
−2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0020	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0126	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0238	.0233
−1.8	.0359	.0352	.0344	.0336	.0329	.0322	.0314	.0307	.0300	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0570	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0722	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
− .9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
− .8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
− .7	.2420	.2389	.2358	.2327	.2297	.2266	.2236	.2206	.2177	.2148
− .6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
− .5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
− .4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3516	.3121
− .3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
− .2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
− .1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
− .0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table I Values of the standard normal distribution function

z	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5363	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7974	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9430	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9648	.9656	.9664	.9671	.9678	.9686	.9693	.9700	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9762	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9874	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.	.9987	.9990	.9993	.9995	.9997	.9998	.9998	.9999	.9999	1.0000

Note 1: If a normal variable X is not "standard," its values must be "standardized": $Z = (X - \mu)/\sigma$. That is, $P(X \leq x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$.

Note 2: For "two-tail" probabilities, see Table Ib.

Note 3: For $z \geq 4$, $\Phi(z) = 1$ to four decimal places; for $z \leq -4$, $\Phi(z) = 0$ to four decimal places.

Note 4: The entries opposite $z = 3$ are for 3.0, 3.1, 3.2, etc.