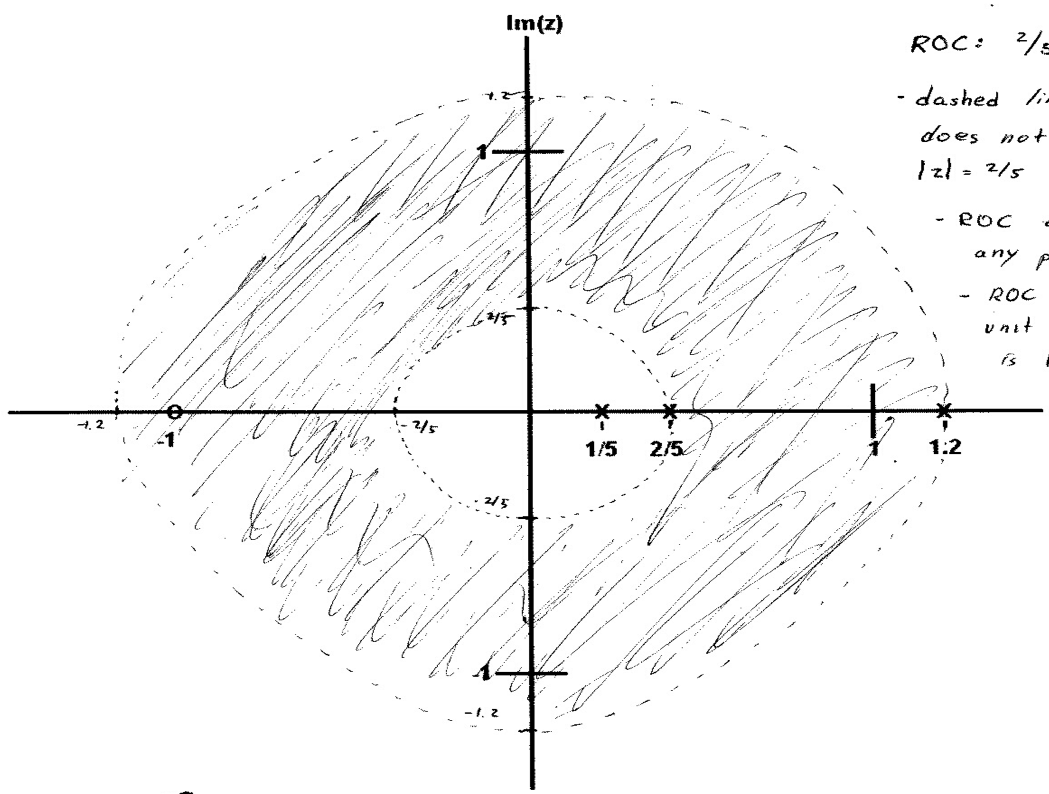
EXAM #2 **Digital Signal Processing (ECE 520)** July 12, 2001	**Name:** SOLUTIONS **Page 1 of 9**

- You will have the full class to complete this exam.
- This exam is closed book, notes, papers, neighbor, etc. Please be sure there are no loose papers around your workspace.
- You will not need a calculator, but if you want to use one, you may use either a 4-function or scientific calculator with *no alphabetic entry* capabilities. *No graphing calculators, PDAs, etc.*
- All paper to be used with the exam will be provided for you. You may not use your own paper.
- If you need more paper or a clarification of a problem, please raise your hand.
- Please be sure you have 9 pages. Please write your name on the first page, and either your name or initials on each subsequent page. On any other paper you hand in with the exam, please write your name in the upper right corner.

GRADE: ________ / 100

Problem #1 (30 points)

10pts a) Given the following pole-zero plot of H(z), the Z-transform of the sequence h[n], and the fact that it is known that the sequence Fourier transform of h[n] converges absolutely, determine if the sequence is left-sided, right-sided, two-sided, or none of the above. **Explain why**, stating *all* reasoning. On the plot, sketch the ROC for H(z) given the above constraints.



ROC: $2/5 < |z| < 1.2$

- dashed line - ROC does not include $|z| = 2/5$ or $|z| = 1.2$.
- ROC cannot contain any poles
- ROC must include the unit circle since it is known that the DTFT exists by absolute convergence

$\boxed{\text{two-sided}}$ The ROC is a ring, therefore, h[n] is two-sided.

5pts b) Assuming the sequence h[n] from part a represents the impulse response of a LTI system, is the system BIBO stable? Is the system causal? **Explain why**, stating *all* reasoning, for both cases.

Because the sequence Fourier transform of h[n] exists by absolute convergence, the system is BIBO stable, as $\sum_{k=-\infty}^{\infty} |h[k]|$ is finite.

The system is not causal, as h[n] is not right-sided, due to the region of convergence, and therefore, $h[n] \neq 0$ for all $n < 0$.

15pts c) Consider a LTI system with an input x[n]=u[n] – u[n – N], N > 0, an impulse response h[n] = α^nu[n], |α| < 1, and an output y[n]. (NOTE: this system should not be confused with the one in parts a and b of this problem).

5pts (i) Find the Z-transform of x[n], using the definition of the Z-transform explicitly. Be sure to state the ROC.

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{N-1} (1) z^{-n} = \underbrace{\sum_{n=0}^{N-1} (z^{-1})^n}_{\text{geometric series}} = \frac{(z^{-1})^0 - (z^{-1})^N}{1 - z^{-1}}$$

$$= \boxed{\frac{1 - z^{-N}}{1 - z^{-1}} = \frac{z - z^{-N+1}}{z - 1} = X(z)}$$

ROC - x[n] is a finite-duration sequence, ∴ the ROC is the entire z-plane, except possibly $z=0$ or $z=\infty$. $z=\infty \Rightarrow X(z)$ finite. $z=0 \Rightarrow X(z)$ undef. ∴ $\boxed{\text{ROC: entire z-plane, except } z = \emptyset}$

5pts (ii) Find the Z-transform of h[n], using the definition of the Z-transform explicitly. Be sure to state the ROC.

$$\mathcal{Z}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \underbrace{\sum_{n=0}^{\infty} (\alpha z^{-1})^n}_{\text{geometric series}} = \frac{1}{1 - \alpha z^{-1}}$$

$$= \boxed{\frac{z}{z - \alpha} = H(z)}$$

ROC – for convergence of the geometric series,

$|\alpha z^{-1}| < 1$

$\boxed{\text{ROC: } |\alpha| < |z|}$

5pts (iii) Find Y(z), the Z-transform of y[n]. Be sure to state the ROC.

$y[n] = x[n] * h[n]$

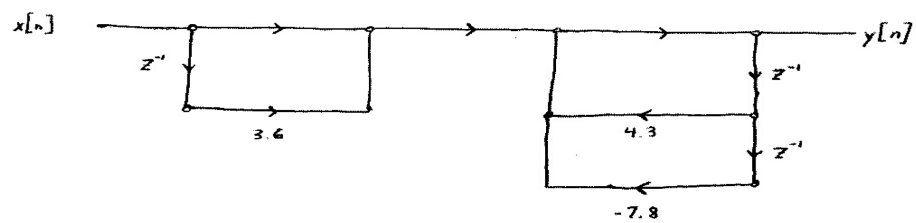
$$Y(z) = X(z)\, H(z) = \boxed{\left(\frac{z - z^{-N+1}}{z - 1}\right)\left(\frac{z}{z - \alpha}\right) = Y(z)}$$

ROC: $R_x \cap R_h$, ∴ $\boxed{\text{ROC: } |z| > |\alpha|}$

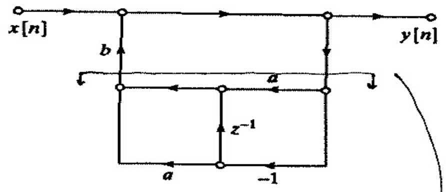
Problem #2 (30 points)

15 pts a) Consider a LTI system with the system function $H(z) = \dfrac{1 + 3.6z^{-1}}{1 - 4.3z^{-1} + 7.8z^{-2}}$. Draw the direct form I implementation. You may use either block diagrams or signal flow graphs. Be sure to clearly label the diagrams.

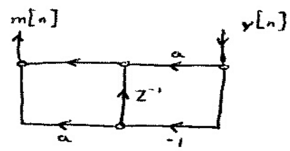
By inspection –



15pts **b)** **Consider the following signal flow graph:**

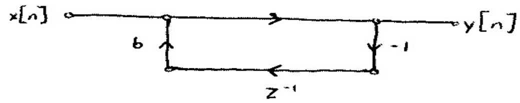


Write a difference equation for y[n]. HINT: you may find it useful to redraw the implementation as a block diagram.



$$m[n] = a\cancel{y[n]} - y[n-1] - a\cancel{y[n]} = -y[n-1]$$

∴ the system can be simplified to:



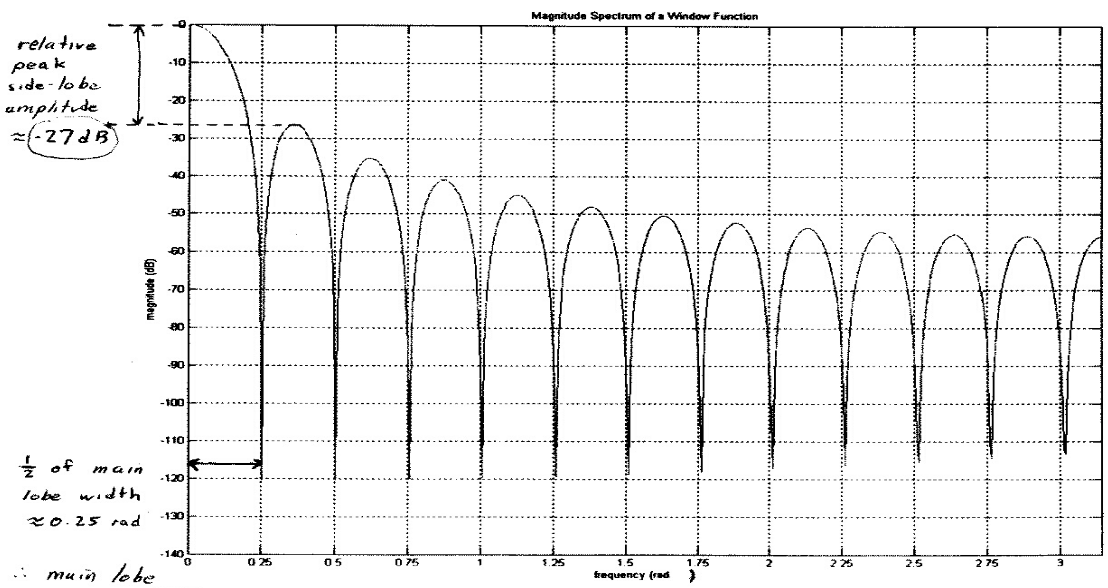
$$\therefore \boxed{y[n] = x[n] - b\,y[n-1]}$$

Problem #3 (10 points)

a) Of the window rectangular, Bartlett, Hanning, Hamming, and Blackman windows, which window function, for a given length, has the narrowest main lobe in its magnitude spectrum, and therefore, the best preservation of discontinuities when windowing $H(e^{j\omega})$ to obtain $H_d(e^{j\omega})$?

The rectangular window

b) The below figure shows the magnitude spectrum of a particular window function, obtained by taking the magnitude of the sequence Fourier transform of the window function. Give approximate values for the relative peak side-lobe amplitude, and the main lobe width. Label these quantities on the plot, showing where they were measured from.



∴ main lobe width ≈ 0.50 rad

c) The rectangular, Bartlett, Hanning, Hamming, and Blackman windows studied all have the property that $w[n] = \begin{cases} w[M-n], 0 \le n \le M \\ 0, else \end{cases}$ (the windows are symmetric about M/2). *Clearly* **explain** what effect this has on the frequency response of a FIR filter designed using one of these windows via the windowing design method, provided that the desired impulse response is also symmetric about M/2. It is not necessary to use equations, but they may help to improve your explanation.

This property allows for the frequency response of the filter to have a generalized linear phase

$$H(e^{j\omega}) = H_e(e^{j\omega})\, e^{-j\omega M/2},$$

where $H_e(e^{j\omega})$ is a real and even function of ω.

Problem #4 (30 points)

a) First, consider an ideal, zero-phase lowpass filter. Give and expression for $H(e^{j\omega})$, the frequency response of the filter, given that the cutoff frequency is ω_c.

over $-\pi < \omega < \pi$

$$\boxed{H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{else} \end{cases}}$$

b) To implement a linear-phase filter, a window function will be used that is symmetric about M/2, where (M+1) is the length of the window function, and is a parameter of the function. Additionally, a modification must be imposed on the ideal frequency response from part a. Impose this restriction, and write an expression for the modified frequency response. This expression will be referred to as $H_{LP}(e^{j\omega})$, the desired frequency response for the filter to implement.

$$\boxed{H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c \\ 0, & \text{else} \end{cases}} \quad \text{over } -\pi < \omega < \pi$$

c) Find an expression for $h_{LP}[n]$, by calculating the inverse sequence Fourier transform of $H_{LP}(e^{j\omega})$ from part b.

$$h_{LP}[n] = \mathfrak{F}^{-1}\{H_{LP}(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega M/2} e^{j\omega n} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j(\frac{M}{2}-n)\omega}\, d\omega$$

$$= \frac{1}{j2\pi(n-\frac{M}{2})}\left[e^{j(n-\frac{M}{2})\omega}\right]_{-\omega_c}^{\omega_c} = \frac{1}{j2\pi(n-\frac{M}{2})}\left[e^{j\omega_c(n-\frac{M}{2})} - e^{-j\omega_c(n-\frac{M}{2})}\right]$$

$$= \boxed{\frac{\sin(\omega_c(n-\frac{M}{2}))}{\pi(n-\frac{M}{2})} = h_{LP}[n]}$$

d) A rectangular window w[n]=u[n]-u[n-M] (M > 0) will be used to window $h_{LP}[n]$ to obtain h[n], the implemented system's impulse response. Give an analytic expression for h [n]. Expand this expression, so that the terms $h_{LP}[n]$ and w[n] do not appear explicitly in your final answer.

$$h[n] = h_{LP}[n]\cdot w[n] = \frac{\sin(\omega_c(n-\frac{M}{2}))}{\pi(n-\frac{M}{2})}\left(u[n] - u[n-M]\right)$$

$$\boxed{h[n] = \begin{cases} \dfrac{\sin(\omega_c(n-\frac{M}{2}))}{\pi(n-\frac{M}{2})}, & 0 \le n < M \\ 0, & \text{else} \end{cases}}$$

e) Now, implement a high pass filter with a cutoff frequency of ω_c, using $H_{LP}(e^{j\omega})$ from part b. Write an expression for $H_{HP}(e^{j\omega})$, the desired frequency response of the implemented high pass filter. Expand this expression, so that the term $H_{LP}(e^{j\omega})$ does not appear explicitly in your final answer.

- $H_{LP}(e^{j\omega})$ was for a linear phase filter, so $H_{HP}(e^{j\omega})$ must be as well.

$$H_{HP}(e^{j\omega}) = e^{-j\omega M/2} - H_{LP}(e^{j\omega}) = \boxed{\begin{cases} 0, & |\omega| < \omega_c \\ e^{-j\omega M/2}, & \omega_c < |\omega| < \pi \end{cases}} \quad \text{- over } -\pi < \omega < \pi$$

f) Find an expression for $h_{HP}[n]$, by calculating the inverse sequence Fourier transform of $H_{HP}(e^{j\omega})$ from part e.

$$h_{HP}[n] = \mathcal{F}^{-1}\{H_{HP}(e^{j\omega})\} = \mathcal{F}^{-1}\{e^{-j\omega M/2}\} - \underbrace{\mathcal{F}^{-1}\{H_{LP}(e^{j\omega})\}}_{\text{found in part (d)}}$$

$$\mathcal{F}^{-1}\{e^{-j\omega M/2}\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega M/2} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(\frac{M}{2}-n)}\, d\omega$$

$$= \frac{-1}{j2\pi(\frac{M}{2}-n)}\left[e^{-j\omega(\frac{M}{2}-n)}\right]_{-\pi}^{\pi} = \frac{1}{j2\pi(n-\frac{M}{2})}\left[e^{j\omega(n-\frac{M}{2})}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{j2\pi(n-\frac{M}{2})}\left[e^{j\pi(n-\frac{M}{2})} - e^{-j\pi(n-\frac{M}{2})}\right] = \frac{\sin(\pi(n-\frac{M}{2}))}{\pi(n-\frac{M}{2})}$$

$$\therefore\ h_{HP}[n] = \frac{\sin(\pi(n-\frac{M}{2})) - \sin(\omega_c(n-\frac{M}{2}))}{\pi(n-\frac{M}{2})}$$