Chapter 4

The Z-Transform

4.1 Introduction

4.1.a The Z-transform

A signal processing system is illustrated in Figure 4.1.

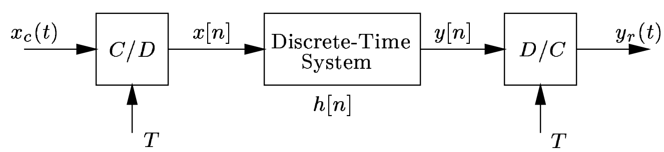


Figure 4.1: Signal Processing System:

Some tools are needed for analysis/synthesis of the discrete-time systems. In particular, these tools should provide capabilities for the study of the system's performance, and system's structure. The Z-transform is of immense value in that regard.

Definition: Given a sequence $x[n]$, $n \in [-\infty, \infty)$, the Z-transform $X(z)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \tag{4.1}$$

where z is a complex parameter. $\diamondsuit$

4.1.b The Inverse Z-transform

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{4.2}$$

where C is a counterclockwise contour that encircles the origin.

We can show that 4.1 and 4.2 provide unique representation for $X(z)$ and $x[n]$. This representation will be denoted by the double-sided arrow:

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{4.3}$$

4.1.c Existence of the Z-transform

$$|X(z)| = |\sum_{n=-\infty}^{\infty} x[n] z^{-n}| \quad < \quad \infty$$

$$\text{If} \qquad \sum_{n=-\infty}^{\infty} |x[n]||z^{-n}| \quad < \quad \infty \tag{4.4}$$

The region in the complex z-domain over which 4.4 is satisfied is called the *region of convergence* (ROC).

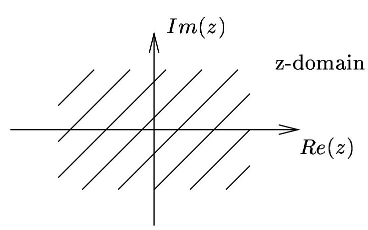


Figure 4.2: Z-domain

Since z is complex, then if 4.4 is satisfied for a value $z = z_1$, it will also be satisfied for all values defined by the circle $|z| = |z_1|$. Therefore, the ROC is a ring in the z-plane centered around the origin, as shown in Figure 4.3.

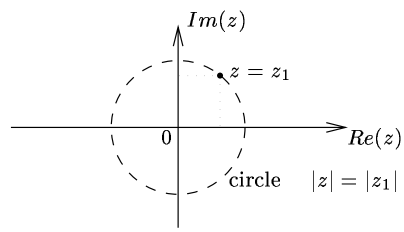


Figure 4.3: Circle $|z| = |z_1|$

4.1.d Relationship between $X(z)$ and $X(f_d)$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (4.5)$$

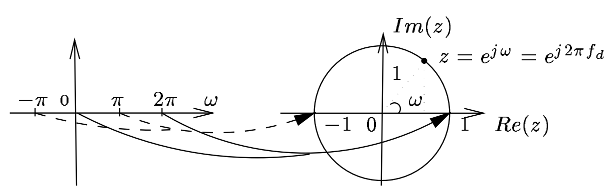
$$X(f_d) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f_d n} \quad (4.6)$$

Therefore,

$$X(f_d) = X(z)|_{z=e^{j2\pi f_d}} \quad (4.7)$$

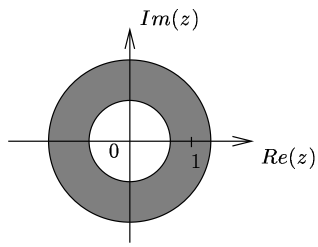
Note:

$z = e^{j2\pi f_d}$ corresponds to the unit-circle in the z-domain. Therefore, if the ROC of the Z-transform includes the unit circle, then the Fourier transform $X(f_d)$ exists, (i.e., $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$). On the other hand, if ROC does not include the unit circle, then $\sum_{n=-\infty}^{\infty} |x[n]|$ does not converge.

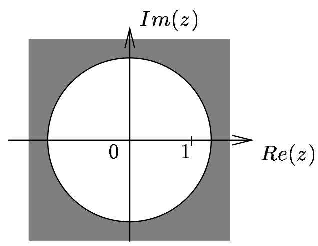


ω	f_d	z
0	0	1
$\pm\pi$	$\pm\frac{1}{2}$	-1
$\pm 2\pi$	± 1	1

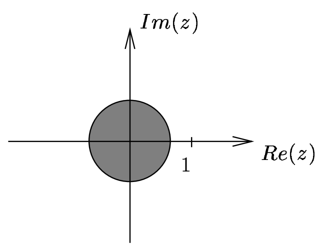
Illustrations:



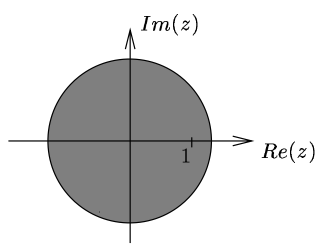
$X(f_d)$ exists since ROC includes the $z = 1$ point.



$X(f_d) = \sum x[n]e^{-j2\pi f_d n}$ does not converge uniformly since ROC does not include $z = 1$ point.



$X(f_d)$ does not converge absolutely



$X(f_d)$ converges absolutely

4.2 Evaluation of the Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

For $X(z)$ to be completely specified, we must identify the ROC for which the Laurent series $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges.

Example 1:

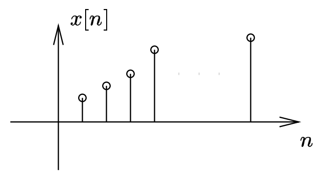
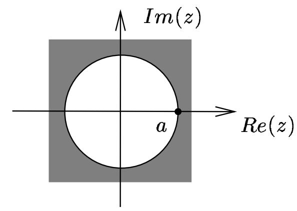


Figure 4.4: Right-sided sequence

$$\begin{aligned} x[n] &= a^n u[n] \\ X(z) &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \\ &\equiv \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \end{aligned}$$

$$\text{if} \quad \begin{aligned} |az^{-1}| &< 1 \\ |a||z^{-1}| &< 1 \\ |a| &< |z| \end{aligned}$$



$$X(z) = \frac{z}{z-a}, \qquad |z| > |a|.$$

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Example 2:

$$x[n] = -a^n u[-n-1]$$

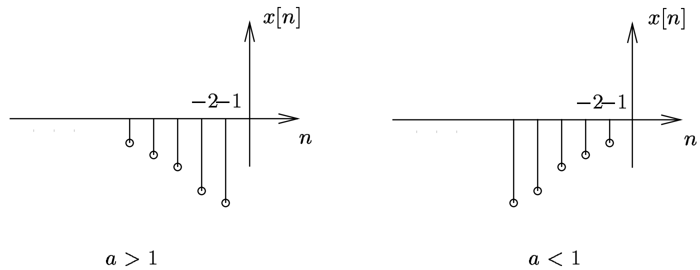
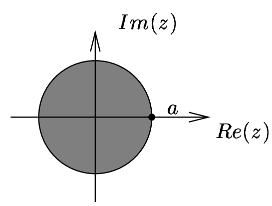


Figure 4.5: Left-sided sequence

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} \\ &\equiv -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n \\ &= -\left(\sum_{n=0}^{\infty} a^{-n} z^n - 1 \right) = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \\ &= 1 - \frac{1}{1 - a^{-1} z} \end{aligned}$$

$$\text{for} \quad \begin{aligned} |a^{-1} z| &< 1 \\ |z| &< a \end{aligned}$$



$$X(z) = \frac{z}{z-a}, \qquad |z| < a$$

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Note:

The sequence $x[n]$ is different in the two examples but $X(z)$ has similar form; the ROCs are *different*.

Example 3:

$$x[n] = \frac{1}{2}^n u[n] + \frac{-1}{3}^n u[n]$$

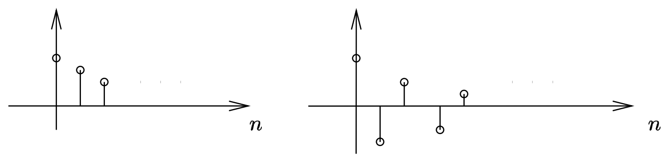

Figure 4.6: Right-sided sequence

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{-1}{3}\right)^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(\frac{-1}{3} z^{-1}\right)^n \\
&= \underbrace{\frac{1}{1 - \frac{1}{2} z^{-1}}}_{|z| > \frac{1}{2}} + \underbrace{\frac{1}{1 + \frac{1}{3} z^{-1}}}_{|z| > \frac{1}{3}} \qquad (4.8)
\end{aligned}
$$

overall ROC must be $|z| > \frac{1}{2}$. Simplifying 4.8, we get

$$
X(z) = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}, \qquad ROC \quad |z| > \frac{1}{2}.
$$

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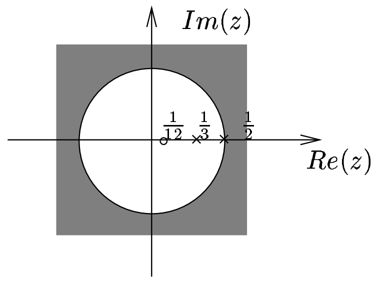
Note:

When writing $X(z)$ as a rational function $X(z) = \dfrac{N(z)}{D(z)}$, the poles of $X(z)$ are the values of z for which $X(z) = \infty$ and the zeros of $X(z)$ are the values of z for which $X(z) = 0$. Symbols for poles are $\times$ and symbols for zeros are $\circ$.



Example 4:

$$
\begin{aligned}
x[n] &= \underbrace{(-1/3)^n u[n]}_{\text{right-sided}} - \underbrace{(1/2)^n u[-n-1]}_{\text{left-sided}} \\
X(z) &= \sum_{n=0}^{\infty}(-\frac{1}{3}z^{-1})^n - \left(1 - \sum_{n=0}^{\infty}(\frac{1}{2}^{-1}z)^n\right) \\
&= \underbrace{\frac{1}{1-(-\frac{1}{3}z^{-1})}}_{|z|>\frac{1}{3}} - \underbrace{\left(1 - \frac{1}{1-\frac{1}{2}^{-1}z}\right)}_{|z|<\frac{1}{2}} \\
&= \underbrace{\frac{1}{1+\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|<\frac{1}{2}}
\end{aligned}
$$

$$\text{Overall ROC: } \frac{1}{3} < |z| < \frac{1}{2}$$

$$X(z) = \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}, \qquad \frac{1}{3} < |z| < \frac{1}{2}$$

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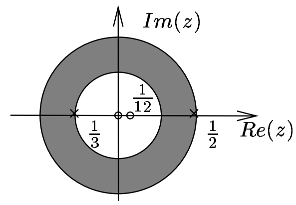
Note:



(1) In Example 1, $x[n]$ was a *right-sided* sequence and the ROC was an *outer* circle.
(2) In Example 2, $x[n]$ was a *left-sided* sequence and the ROC was an *inner* circle.
(3) In Example 4, $x[n]$ has a *right-sided* and *left-sided* sequence, and the ROC was a *ring*.

Example 5:

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{elsewhere} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-(az^{-1})}$$

$$\text{ROC:} \quad |az^{-1}| < \infty, \quad |a| < \infty, \quad |z| \neq 0$$

$$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}, \qquad |a| < \infty, \quad |z| \neq 0$$

Zeros occur when $z^N = a^N$, where $z_K = ae^{j2\pi k/N}$ for $k = 0, 1, 2, ..., N-1$. The poles are at $z = 0$, since the pole at $z = a$ is canceled by the zero at $z = a$; i.e., for $k = 0$. Therefore, the ROC of this system includes the entire z-plane except at $z = 0$.

$\diamond \diamond \diamond$

Example 6:

$$\begin{aligned} x[n] &= \delta[n] \\ X(z) &= \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1 \end{aligned}$$

ROC: all z

$\diamond \diamond \diamond$

Example 7:

$$\begin{aligned} x[n] &= u[n] \\ X(z) &= \sum_{n=0}^{\infty} z^{-n} \\ &= \frac{1}{1-z^{-1}} = \frac{z}{z-1} \end{aligned}$$

$$\text{ROC:} \quad |z^{-1}| < 1, \qquad |z| > 1$$

$\diamond \diamond \diamond$

4.3 Properties of the ROC for the Z-Transform

The examples of the previous section suggest that the properties of the region of convergence depend on the nature of the signal. These properties are summarized below, followed by some discussion and intuitive justification. We assume specifically that the algebraic expression for the z-transform is a rational function and that $x[n]$ has finite amplitude except possibly at $n = \infty$ or $n = -\infty$.

> PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin, i.e., $0 \leq r_R < |z| < r_L \leq \infty$.

> PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely *if and only if* the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z-plane except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is a n infinity-duration sequence that is neither right-sided nor left-sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole, and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

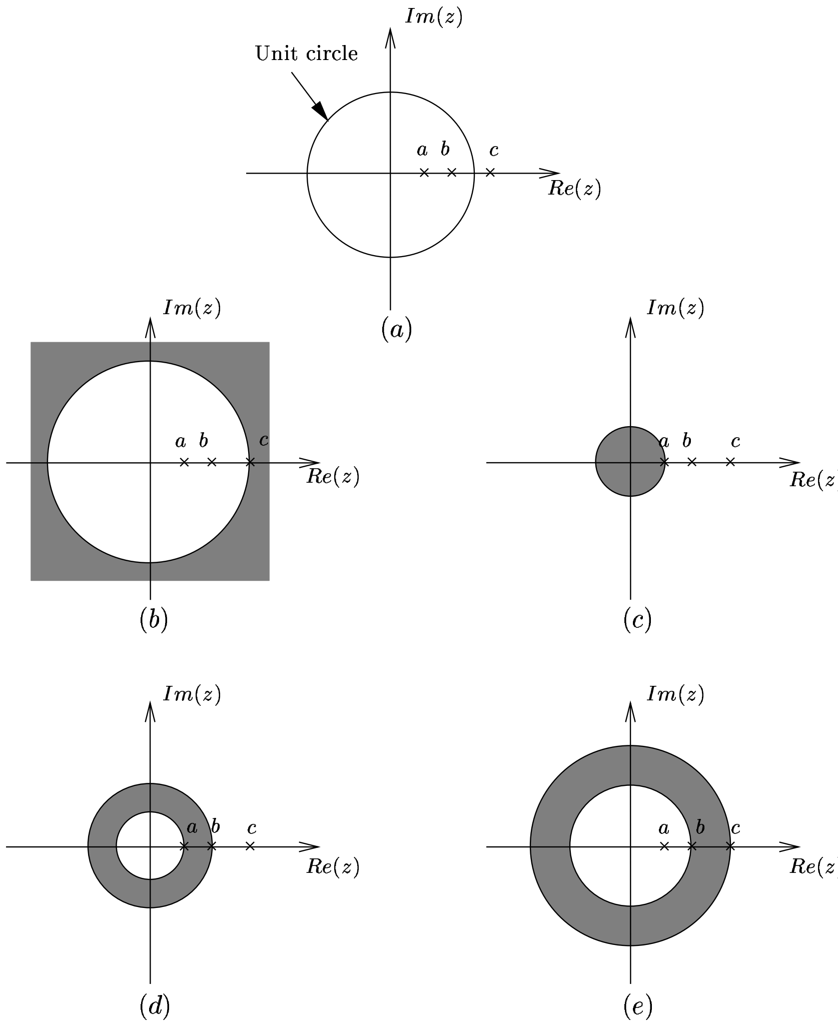


Figure 4.7: Examples of four z-transforms with the same pole-zero locations, illustrating the different possibilities for the region of convergence. Each corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) To a two-sided sequence.

4.4 Properties of the Z-transform

(i) LINEARITY

$$
\begin{aligned}
x_1[n] &\longleftrightarrow X_1(z); \qquad \text{ROC} = R_{x_1} \\
x_2[n] &\longleftrightarrow X_2(z); \qquad \text{ROC} = R_{x_2} \\
ax_1[n] + bx_2[n] &\longleftrightarrow aX_1(z) + bX_2(z); \qquad \text{ROC} = R_{x_1} \cap R_{x_2}
\end{aligned}
$$

(ii) TIME SHIFTING

$$
\begin{aligned}
x[n] &\longleftrightarrow X(z); \qquad \text{ROC} = R_x \\
x[n-n_o] &\longleftrightarrow z^{-n_o}X(z); \qquad \text{ROC} = R_x
\end{aligned}
$$

where the ROC may have possible addition or deletion of $z = 0$ or $z = \infty$.

Proof:

$$
\begin{aligned}
Z\left(x[n-n_o]\right) &\triangleq \sum_{n=-\infty}^{\infty} x[n-n_o]z^{-n}, \qquad \text{let} m = n - n_o \\
&= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_o)} \\
&= z^{-n_o} \sum_{m=-\infty}^{\infty} x[m]z^{-m} \\
&= z^{-n_o}X(z)
\end{aligned}
$$

If $n_o > 0$, the number of poles at $z = 0$ in the ROC may change. If $n_o < 0$, the number of zeros at $z = 0$ in the ROC may change. For all other cases ROC$= R_x$.

Example 8:

$$
\begin{aligned}
u[n] &\longleftrightarrow \frac{z}{z-1}; \qquad |z| > 1 \\
u[n-1] &\longleftrightarrow z^{-1}\frac{z}{z-1} = \frac{1}{z-1}, \qquad |z| > 1
\end{aligned}
$$

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Example 9:

$$
\begin{aligned}
\delta[n] &\longleftrightarrow 1; \qquad \text{ROC is all } z \\
\delta[n-n_o] &\longleftrightarrow z^{-n_o}
\end{aligned}
$$

where the ROC is all z except for multiple poles at $z = 0$ if $n_o > 0$.

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(iii) Multiplication by an Exponential Sequence

$$\begin{aligned} x[n] &\longleftrightarrow X(z); \qquad \text{ROC} = R_x \\ a^n x[n] &\longleftrightarrow X(\frac{z}{a}); \qquad \text{ROC} = |a| R_x \end{aligned}$$

Proof:

$$\begin{aligned} Z\left(a^n x[n]\right) &\triangleq \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n] (\frac{z}{a})^{-n} \\ &\triangleq X(\frac{z}{a}) \end{aligned}$$

The ROC will be substituting $(\frac{z}{a})$ for z in R_x. Hence, the ROC equals $|a|R_x$.

Example 10:

$$\begin{aligned} a^n u[n] &= x[n] \\ u[n] &\longleftrightarrow \frac{z}{z-1}, \qquad \text{ROC: } |z| > 1 \\ a^n u[n] &\longleftrightarrow \frac{z/a}{z/a - 1} = \frac{z}{z-a}, \qquad \text{ROC: } |z/a| > 1, \quad |z| > |a| \end{aligned}$$

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(iv) Differentiation

$$\begin{aligned} x[n] &\longleftrightarrow X(z); \qquad \text{ROC} = R_x \\ n x[n] &\longleftrightarrow -z \frac{d}{dz} X(z) \end{aligned}$$

where the ROC$= R_x$, except with possible addition or deletion of $z = 0$ or $z = \infty$.

Proof:

$$X(z) = Z\left(x[n]\right) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Differentiating both sides with respect to z,

$$\begin{aligned}\frac{d}{dz}X(z) &= \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n}) \\ &= -\sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1} \\ &= -z^{-1}\sum_{n=-\infty}^{\infty} [nx[n]]z^{-n}\end{aligned}$$

Rearranging,

$$\begin{aligned}-z\frac{d}{dz}X(z) &= \sum_{n=-\infty}^{\infty} (nx[n])\, z^{-n} \\ nx[n] &\longleftrightarrow -z\frac{d}{dz}X(z)\end{aligned}$$

Example 11:

$$u[n] \longleftrightarrow \frac{z}{z-1}; \qquad \text{ROC: } |z|>1$$

$$nu[n] \longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left[\frac{(z-1)-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$\text{ROC: } |z|>1$$

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(v) Conjugate of a Complex Sequence

$$\begin{aligned}x[n] &\longleftrightarrow X(z); \qquad \text{ROC} = R_x \\ \overline{x}[n] &\longleftrightarrow \overline{X}(\overline{z}), \qquad \text{ROC} = R_x\end{aligned}$$

Proof:

$$\begin{aligned}Z\left(\overline{x}[n]\right) &\triangleq \sum_{n=-\infty}^{\infty} \overline{x}[n]z^{-n} \\ &= \overline{\sum_{n=-\infty}^{\infty} x[n](\overline{z})^{-n}} \\ &\triangleq \overline{X}(\overline{z}), \quad \text{ROC} = R_x\end{aligned}$$

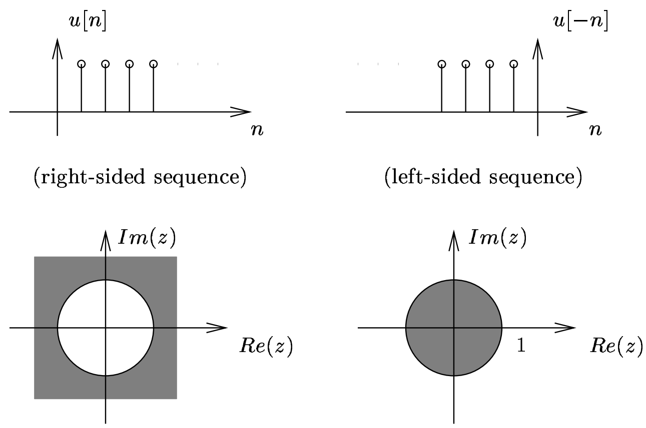
(vi) TIME REVERSAL

$$\begin{aligned} x[n] &\longleftrightarrow X(z): && \text{ROC} = R_x \\ x[-n] &\longleftrightarrow X\left(\frac{1}{z}\right); && \text{ROC} = \frac{1}{R_x} \end{aligned}$$

Proof: Use definition

Example 12:

$$\begin{aligned} u[n] &\longleftrightarrow \frac{z}{z-1}, && |z| > 1 \\ u[-n] &\longleftrightarrow \frac{1/z}{1/z-1} = \frac{1}{1-z}, && |z| < 1 \end{aligned}$$



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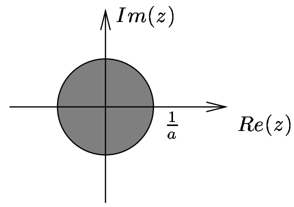
Example 13:

$$\begin{aligned} x[n] &= a^{-n}u[-n] \\ X(z) &= \frac{1}{1-az}, && |z| < |1/a| \end{aligned}$$



compare with $x[n] = a^{-n}u[n]$.

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(vii) Convolution of Sequences

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z); \qquad \text{ROC} = R_{x_1} \cap R_{x_2}$$

Proof:

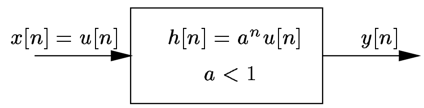
$$\begin{aligned} \text{Let} \qquad y[n] &= x_1[n] * x_2[n] \triangleq \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \\ Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] \left(\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \right) \end{aligned}$$

Let $n - k = m$,

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \\ &\triangleq X_1(z)X_2(z) \end{aligned}$$

The ROC is the product (intersection) of R_{x_1} and R_{x_2}

Example 14:



$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
X(z) &\longleftrightarrow \frac{z}{z-1}; \qquad |z| > 1 \\
H(z) &\longleftrightarrow \frac{z}{z-a}; \qquad |z| > a \\
Y(z) &= \frac{z^2}{(z-1)(z-a)}; \qquad |z| > 1
\end{aligned}
$$

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(viii) Initial Value Theorem

$$x[0] = \lim_{z \to \infty} X(z)$$

The previous properties are shown in Figure 4.8.

4.5 The Inverse Z-Transform

$$
\begin{aligned}
x[n] &\longleftrightarrow X(z) \\
X(z) &\stackrel{\triangle}{=} \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
x[n] &= \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz.
\end{aligned}
$$

Evaluation of the contour integral is not always trivial. Common Z-transform pairs have been tabulated. $X(z)$ has to be manipulated to reach a form similar to the tabulated pairs; also the properties of the Z-transform are often used in this process. We'll discuss few approaches to evaluate the inverse Z-transform.

Sequence	**Transform**	**ROC**
1. $x[n]$	$X(z)$	R_x
2. $x_1[n]$	$X_1(z)$	R_{x_1}
3. $x_2[n]$	$X_2(z)$	R_{x_2}
4. $ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
5. $x[n-n_o]$	$z^{-n_o}X(z)$	R_x except for the possible addition or deletion of the origin or ∞
6. $z_o^n x[n]$	$X(z/z_o)$	$\|z_o\|R_x$
7. $nx[n]$	$-z\frac{dX(z)}{dz}$	R_x except for the possible addition or deletion of the origin or ∞
8. $x^*[n]$	$X^*(z^*)$	R_x
9. $\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
10. $\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
11. $x[-n]$	$X(1/z)$	$1/R_x$
12. $x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
13. $x[n]=0, \quad n<0$ $\lim_{z\to\infty} X(z) = x[0]$		
14. $x_1[n]x_2[n]$	$\frac{1}{2\pi j}\oint_c X_1(v)X_2(z/v)v^{-1}dv$	Contains $R_{x_1}R_{x_2}$
15. $\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint X_1(v)X_2^*(1/v^*)v^{-1}dv$		

Figure 4.8: Some Z-transform Properties

(1) By Inspection/Tables

Example 15:

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}, \qquad |z| > a$$

$x[n]$ for any variation(value) of a could easily be deduced. For example, if

$$\begin{aligned} X(z) &= \frac{5}{3-z^{-1}}, \qquad |z| > 1/3 \\ &= \frac{5/3}{1-\frac{1}{3}z^{-1}}, \qquad |z| > 1/3 \\ x[n] &= \frac{5}{3}\left(\frac{1}{3}\right)^n u[n] \end{aligned}$$

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(2) Partial Fractions and Tables

Suppose $X(z)$ can be expressed as a rational polynomial in z^{-1} of the form

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{i=0}^{N} a_i z^{-i}} \triangleq \frac{N(z)}{D(z)} \tag{4.9}$$

Suppose also that both $N(z)$ and $D(z)$ polynomials can be factorized; i.e.,

$$X(z) = \frac{b_o \Pi_{k=1}^{M}\left(1-c_k z^{-1}\right)}{a_o \Pi_{i=1}^{N}\left(1-d_i z^{-1}\right)} \tag{4.10}$$

The above form can be decomposed into a number of lower order terms, for which a tabular Z-transform may exist. We will consider the following cases:

(i) $X(z)$ is proper; i.e. $M < N$.

(ii) $X(z)$ is improper; i.e., $M \geq N$

(i) X(z) is strictly proper ($M < N$)
Case 1: $D(z)$ has simple poles.

In this case, $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})} \tag{4.11}$$

where,

$$A_k = \lim_{z \to d_k} (1 - d_k z^{-1}) X(z) \tag{4.12}$$

Example 16:

$$\begin{aligned} X(z) &= \frac{1}{(1 - z^{-1})(1 - az^{-1})}, \qquad |z| > 1 \\ &= \frac{A_1}{1 - z^{-1}} + \frac{A_2}{(1 - az^{-1})} \end{aligned}$$

Where,

$$\begin{aligned} A_1 &= \lim_{z \to 1}(1 - z^{-1})X(z) \equiv \left(\frac{1}{1 - az^{-1}} \right) |_{z=1} \\ &= \frac{1}{1 - a} \\ A_2 &= \lim_{z \to a}(1 - az^{-1})X(z) \equiv \left(\frac{1}{1 - z^{-1}} \right) |_{z=a} \\ &= \frac{a}{a - 1} \end{aligned}$$

$$\begin{aligned} X(z) &= \frac{1/(1 - a)}{1 - z^{-1}} + \frac{a/(a - 1)}{1 - az^{-1}}, \qquad |z| > 1 \\ x[n] &= \frac{1}{1 - a} u[n] + \left(\frac{a}{a - 1} \right) a^n u[n] \end{aligned}$$

Case 2: $D(z)$ has multiple poles.

Suppose,

$$\begin{aligned} X(z) &= \frac{N(z)}{D(z)} \\ &= \frac{N(z)}{(1 - d_1 z^{-1})(1 - d_2 z^{-1})^m ... (z^{-2} + \gamma z^{-1} + \beta)} \end{aligned}$$

where m is the order of the pole at $z = d_2$ and γ and β are constants. In this case, we can write $X(z)$ as follows:

$$X(z) = \frac{A_1}{(1-d_1z^{-1})} + \sum_{m=1}^{s} \frac{B_m}{(1-d_2z^{-1})^m} + \frac{Cz^{-1}+D}{z^{-2}+\gamma z^{-1}+\beta} \tag{4.13}$$

where

$$\begin{aligned} A_1 &= \lim_{z\to d_1}(1-d_1z^{-1})X(z) \\ B_m &= \lim_{W\to d_2^{-1}} \frac{1}{(s-m)!(-d_2)^{s-m}} \left\{ \frac{d^{s-m}}{dW^{s-m}}(1-d_2W)^s X(W^{-1}) \right\} \end{aligned}$$

C and D are obtained by equating the coefficients.

Example 17:

$$\begin{aligned} X(z) &= \frac{1}{(1+\frac{1}{2}z^{-1})^2(1-2z^{-1})(1-3z^{-1})}, \qquad \frac{1}{2} < |z| < 2 \\ &= \sum_{m=1}^{2} \frac{B_m}{(1+\frac{1}{2}z^{-1})^m} + \frac{A_1}{1-2z^{-1}} + \frac{A_2}{1-3z^{-1}} \\ B_1 &= \lim_{W\to -2} \frac{1}{1!\left(\frac{1}{2}\right)^1} \frac{d}{dW} \frac{1}{(1-2W)(1-3W)} \\ &= \lim_{W\to -2} \frac{2[-12W+5]}{[(1-2W)(1-3W)]^2} = \frac{58}{1225} \\ B_2 &= \lim_{W\to -2} \frac{1}{(1-2W)(1-3W)} = \frac{1}{35} \\ A_1 &= \lim_{z\to 2} \frac{1}{(1+\frac{1}{2}z^{-1})(1-3z^{-1})} = \frac{-1568}{1225} \\ A_2 &= \frac{2700}{1225} \end{aligned}$$

$$x[n] = \frac{58}{1225}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{35}(n+1)\left(-\frac{1}{2}\right)^{n+1} u[n+1] - \frac{1568}{1225}2^n u[-n-1] + \frac{2700}{1225}3^n u[-n-1]$$

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(ii) $X(z)$ is improper; i.e., $M \geq N$.

In this case, we may be able to write $X(z)$ as

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \tag{4.14}$$

where B_r is obtained by long-division and A_k is obtained as before.

$$\begin{aligned} X(z) &= \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \qquad |z| > 1 \\ &= \frac{1 + 2z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}, \qquad |z| > 1 \\ &= B_o + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}} \end{aligned}$$

After using long division,

$$\begin{aligned} X(z) &= 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \\ A_1 &= \lim_{z \to 2} \frac{(1 - \frac{1}{2}z^{-1})(-1 + 5z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \\ &= 9 \\ A_2 &= 8 \end{aligned}$$

Therefore,

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}} \tag{4.15}$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]. \tag{4.16}$$

(3) Power Series Expansion Method

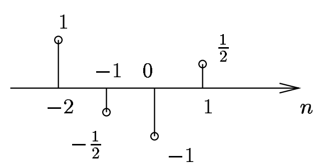
$$\begin{aligned} X(z) &\triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= ...x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2}... \end{aligned}$$

where the sequence $x[n]$ is formed of the following coefficients: $...x[-2], x[-1], x[0], x[1], x[2], ...$ Based on these coefficients, a closed form solution for $x[n]$ may be obtained.

Example 18:

$$\begin{aligned} X(z) &= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1} \\ &= x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} \end{aligned}$$

$$\begin{aligned} x[-2] &= 1 \\ x[-1] &= -\frac{1}{2} \\ x[0] &= -1 \\ x[1] &= \frac{1}{2} \end{aligned}$$



$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$$

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Example 19:

Determine the inverse Z-transform by any method

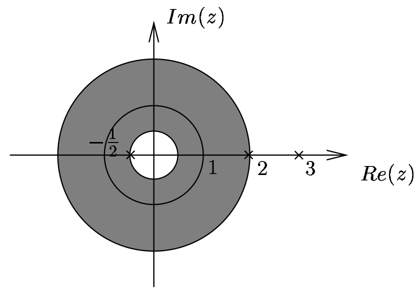
(a) $X(z) = \dfrac{1}{(1+\frac{1}{2}z^{-1})^2(1-2z^{-1})(1-3z^{-1})}$; Stable sequence

(b) $X(z) = e^{z^{-1}}$;

(c) $X(z) = \dfrac{z^3 - 2z}{z-2}$; Left-sided sequence

Solution

(a) Because the sequence is stable, the ROC includes the unit circle.



$$X(z) = \frac{B_2}{(1+\frac{1}{2}z^{-1})^2} + \frac{B_1}{(1+\frac{1}{2}z^{-1})} + \frac{C}{(1-2z^{-1})} + \frac{D}{(1-3z^{-1})} \tag{4.17}$$

(b)

$$\begin{aligned} X(z) &= e^{z^{-1}} \\ &\equiv 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + ... \\ &\triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n} \end{aligned}$$

By comparing the two expressions, we get:

$$
\begin{aligned}
x[0] &= 1 = \frac{1}{0!} \\
x[1] &= 1 = \frac{1}{1!} \\
x[2] &= \frac{1}{2!} \\
x[3] &= \frac{1}{3!} \\
&\cdot \\
&\cdot \\
&\cdot \\
x[n] &= \frac{1}{n!}u[n]
\end{aligned}
$$

(c)

$$X(z) = \frac{z^3 - 2z}{z - 2}; \qquad \text{Left-sided sequence}$$

From long division:

$$
\begin{aligned}
X(z) &= z^2 + 2z + \frac{2z}{z-2} \\
&= z^2 + 2z + \frac{2}{1 - 2z^{-1}}
\end{aligned}
$$

where

$$
\begin{aligned}
x[-2] &= 1 \\
x[-1] &= 2 \\
|z| &< 2
\end{aligned}
$$

$$x[n] = \delta[n+2] + 2\delta[n+1] - 2(2^n)u[-n-1] \tag{4.18}$$

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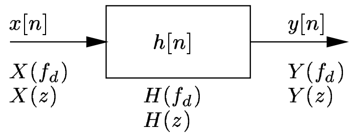
4.6 Transform Analysis of LTI Systems

Consider a LTI system with input $x[n]$, output $y[n]$, and impulse response $h[n]$.

Let $X(f_d)$,$H(f_d)$, and $Y(f_d)$ be the sequence Fourier transforms, and $X(z)$, $H(z)$, and $Y(z)$ be the Z-transforms of $x[n]$, $h[n]$, and $y[n]$, respectively.



4.6.a Frequency Response

$$Y(f_d) = H(f_d)X(f_d) \tag{4.19}$$

is, in general, complex. Therefore, we can define the *magnitude* and the *phase response* of the system as

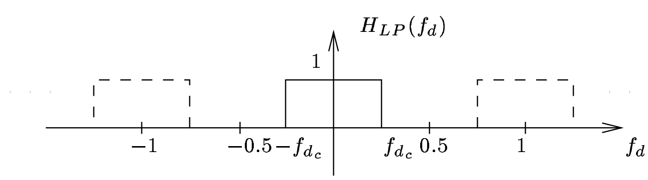
$$|Y(f_d)| = |H(f_d)X(f_d)| \equiv |H(f_d)||X(f_d)|, \tag{4.20}$$
$$ang \ \ Y(f_d) = ang \ \ H(f_d) + ang \ \ X(f_d) \tag{4.21}$$

Note:

$Y(f_d)$, $|Y(f_d)|$, and $ang \ \ Y(f_d)$ are periodic in f_d with a period of 1.

Example 20: Ideal Lowpass Filter

$$H_{LP}(f_d) = \begin{cases} 1, & |f_d| < f_{d_c} \\ 0, & f_{d_c} < |f_d| \leq \frac{1}{2} \end{cases} \tag{4.22}$$
$$h_{lp}[n] = \frac{sin 2\pi f_c n}{\pi n} \qquad -\infty < n < \infty \tag{4.23}$$



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Note:

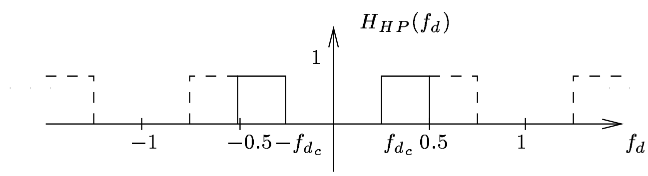
- Ideal LPF has *zero phase*, otherwise, it will have phase distortion.
- Ideal LPF is non-realizable (non-causal)
- Approximations to ideal LPF should have zeros phase also.

Example 21: Ideal Highpass Filter

$$H_{HP}(f_d) = \begin{cases} 0, & |f| < f_{d_c} \\ 1, & f_{d_c} < |f_d| \leq \frac{1}{2} \end{cases} \tag{4.24}$$

$$h_{hp}[n] = \delta[n] - h_{lp}[n] \tag{4.25}$$

$$= \delta[n] - \frac{sin 2\pi f_c n}{\pi n} \tag{4.26}$$



Phase Distortion and Group Delay

Recall that

$$\text{If} \quad x[n] \longleftrightarrow X(f_d), \quad \text{then}$$
$$x[n-d] \longleftrightarrow e^{-j2\pi f_d d} X(f_d),$$

that is, a delay d in the sequence provides a phase shift of $-j2\pi f_d d$ in the frequency response. Therefore, the maximum that a phase shift of the form $e^{\pm j2\pi f_d d}$ can do to the original sequence is to induce time delay, and vice-versa; a time-delay in $x[n]$ by $\pm d$ induces a phase shift in $H(f_d)$ by $e^{\mp j2\pi f_d}$. The phase response will be

$$ang \quad H_d(f_d) = -j2\pi f_d + ang \quad H(f_d) \tag{4.27}$$

that is, the induced phase shift effect is linear in f_d and it is the best type of *phase distortion* (deviation from ideal case) that we can hope for. A measure of phase distortion that is widely used is the so-called *group delay*, $\tau(f_d)$, which is defined as

$$\tau_g(f_d) = -\frac{1}{2\pi}\frac{d}{df_d}\left\{ang \quad H(f_d)\right\}. \tag{4.28}$$

It provides a measure of nonlinearity of phase (deviation from linearity).

Example 22:

$$\begin{aligned}
H_{IdealLPF}(f_d) &= \left\{ \begin{array}{ll} 1, & |f_d| < f_{d_c} \\ 0, & |f_{d_c}| < |f_d| \leq \frac{1}{2} \end{array} \right\} \\
ang \quad H(f_d) &= 0 \\
\tau_g(f_d) &= 0 \\
H_{LP}(f_d) &= \left\{ \begin{array}{ll} e^{-j2\pi f_d d}, & |f_d| < f_{d_c} \\ 0, & f_{d_c} < |f_d| \leq \frac{1}{2} \end{array} \right. \\
\tau_g(f_d) &= -\frac{1}{2\pi}\frac{d}{df_d} ang \quad H_{LP}(f_d) \\
&= -\frac{1}{2\pi}\frac{d}{df_d}(-j2\pi f_d d) \\
&= d \\
h_{lp}[n] &= \frac{\sin 2\pi f_{d_c}(n-d)}{\pi(n-d)}
\end{aligned}$$

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Phase Delay

$$\tau_p(f_d) \triangleq -\frac{1}{2\pi f_d} ang \quad H(f_d) \tag{4.29}$$

e.g., in Example 22,

$$\begin{aligned}
\tau_p(f_d) &= -\frac{1}{2\pi f_d}(-2\pi f_d d) \\
&= d \equiv \tau_g(f_d)
\end{aligned}$$

4.6.b LTI Systems Represented by Linear Constant Coefficients Difference Equations

Consider the class of LTI systems in which the input-output relationship is given by a *linear* (i.e., no $y[n]$ or $x[n]$ terms raised to a power other than 1) *Constant Coefficients* (i.e., no factors such as n, $\sqrt{n}$, ... etc.) difference equation of the form:

$$\sum_{k=0}^{N} a_K y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{4.30}$$

We show that in such systems the Z-transform provides a powerful tool in their study. In particular, issues such as causality, stability, and system structure can be readily studied using the Z-transform which is a generalization of the sequence Fourier transform. Taking the Z-transform of the two sides of 4.30, we get

$$\begin{aligned} \sum_{k=0}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\ \frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \triangleq H(z), \end{aligned} \tag{4.31}$$

where $H(z)$ is denoted by the system function.

Notes:

(i) Various ROC can be associated with 4.31, therefore, the difference equation representation is not unique; i.e., various systems can have the same difference equation.

(ii) Only general requirements of ROC can be assumed; i.e., its a ring and contains no poles.

(iii) If the system is assumed to be *causal*, then the ROC is outside the outermost pole to produce a right-sided sequence of $h[n]$.

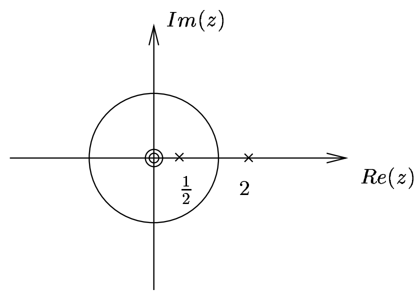
(iv) If the system is stable, then the ROC must contain the unit circle.

(v) From (iii) and (iv), *causality* and *stability* might be opposing or non compatible characteristics of the LTI system.

(vi) For the system to be both *causal* and *stable* all the poles *must* be within the unit circle.

Example 23:

$$\begin{aligned} y[n] &- \frac{5}{2}y[n-1] + y[n-2] = x[n] \\ H(z) &= \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} \\ &= \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \end{aligned}$$



The possibilities of ROC:

The possibilities of ROC: (i) $\frac{1}{2} < |z| < 2$
(ii) $|z| > 2$
(iii) $|z| < \frac{1}{2}$

- For the system to be causal, ROC: $|z| > 2$
- For the system to be stable, ROC: $\frac{1}{2} < |z| < 2$
- The system is neither stable nor causal when the ROC: $|z| < \frac{1}{2}$

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Inverse Systems

From 4.31,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \triangleq \frac{N(z)}{D(z)}$$

Suppose we can factorize both $N(z)$ and $D(z)$; i.e., we can write

$$H(z) = \frac{b_o \Pi_{k=1}^{M}(1 - c_k z^{-1})}{a_o \Pi_{k=1}^{N}(1 - d_k z^{-1})} \tag{4.32}$$

We can define another function $H_i(z)$ such that

$$H(z)H_i(z) = 1 \tag{4.33}$$

Therefore,

$$H_i(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{\sum_{k=0}^{M} b_k z^{-k}} \tag{4.34}$$

and if $H(z)$ is factorized as in 4.32, then

$$H_i(z) = \frac{a_o \Pi_{k=1}^{N}(1 - d_k z^{-1})}{b_o \Pi_{k=1}^{M}(1 - c_k z^{-1})} \tag{4.35}$$

the system which satisfies 4.33 - 4.35 is known as the *inverse system.*

Notes:

(i) From 4.33, the ROC of $H_i(z)$ and $H(z)$ must overlap

(ii) If $H(z)$ and $H_i(z)$ can be as in 4.32 and 4.34, then if the system is causal, the ROC of $H(z)$ is $|z| > \max_k(d_k)$ and $H_i(z)$ has arbitrary ROC but overlaps $|z| > \max_k(d_k)$.

Example 24:

$$\begin{aligned} H(z) &= \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}} \qquad \text{with ROC: } |z| > 0.9 \\ H_i(z) &= \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}} \end{aligned}$$

The possibilities of ROC for $H_i(z)$:

(i)	Inside pole at $z = 0.5$; i.e., $\|z\| < 0.5$
(ii)	Outside pole at $z = 0.5$; i.e., $\|z\| > 0.5$
(iii)	Has to overlap ROC of $H(z)$

Therefore, the ROC of $H_i(z)$ is such that $|z| > 0.5$.

- Because the ROC: $|z| > 0.9$ outside of $z = 0.5$, then the system $H_i(z)$ is *causal.*
- Because the ROC: $|z| > 0.9$ includes the unit circle, then the system $H_i(z)$ is *stable.*

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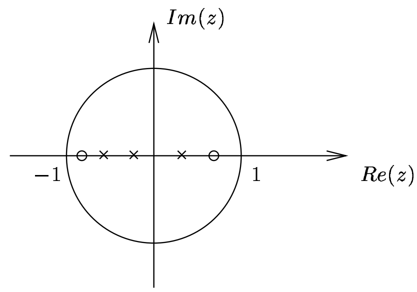
Fact:

A LTI system is *stable* and *causal* and also has a *stable* and *causal* inverse system *if and only if* the poles and zeros of $H(z)$ are inside the unit circle.



Example 25:

$$H(z) = \frac{0.5(1-0.6z^{-1})(1-0.9z^{-1})}{(1-0.2z^{-1})(1-0.3z^{-1})(1-0.85z^{-1})}$$

is the system function for a *causal* and *stable* LTI system - Likewise $H_i(z)$.

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4.7 Infinite Impulse Response (IIR) and Finite Impulse Response (FIR) Systems

Consider $H(z)$ in 4.32; i.e.,

$$H(z) = \frac{b_o \Pi_{k=1}^{M}(1-c_k z^{-1})}{a_o \Pi_{k=1}^{N}(1-d_k z^{-1})} = \frac{N(z)}{D(z)}$$

We can, using partial fractions, simplify $H(z)$ to

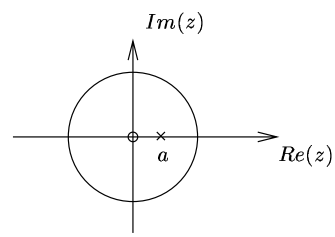
$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{(1-d_k z^{-1})}, \tag{4.36}$$

where B_r is obtained by long division of $N(z)$ by $D(z)$ and $A_k = \lim_{z\to d_k}(1-d_k z^{-1})H(z)$. Recall that B_r terms will be present only if $M \geq N$. Now we can define two types of LTI systems for which $h[n]$ will be distinctly different.

4.7.a IIR Systems

An IIR system is a system in which $h[n]$ is *infinitely long* and will result if at least one pole in $H(z)$ will *not* be canceled by a zero.
Example 26:

$$\begin{aligned} y[n] &- ay[n-1] = x[n] \\ H(z) &= \frac{1}{1-az^{-1}} = \frac{z}{z-a} \end{aligned}$$



where the ROC: $|z| > a$.

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Note:

- pole at $z = a$ is not canceled by the zero at $z = 0$.
- From the tables, $h[n] = a^n u[n]$

4.7.b FIR Systems

An FIR system has *no poles* except at $z = 0$; i.e., $N = 0$ in $H(z)$; that is:

$$H(z) = b_o \Pi_{k=1}^{M}(1 - c_k z^{-1}) \tag{4.37}$$

$$= \sum_{k=0}^{M} b_k z^{-k} \tag{4.38}$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \equiv \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{else} \end{cases} \tag{4.39}$$

The difference equation will be:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \tag{4.40}$$

Example 27:

$$h[n] = \begin{cases} a^n, & 0 \le n < M \\ 0, & \text{else} \end{cases} \tag{4.41}$$

$$H(z) = \sum_{n=0}^{M} a^n z^{-n}$$

$$= \sum_{n=0}^{M} (az^{-1})^n = \frac{1-(az^{-1})^{M+1}}{1-az^{-1}} \triangleq \frac{N(z)}{D(z)} \tag{4.42}$$

$$N(z) = 1-(az^{-1})^{M+1} = \Pi_{k=0}^{M}(z - ae^{\frac{j2\pi k}{(M+1)}})$$

therefore, $H(z)$ has no poles since the zero at $z = a$ will cancel the pole at $z = a$.

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Notes:

(1) Solution of $1 - x^n = 0$ is $x = e^{\frac{j2\pi k}{M}}, k = 0, 1, 2, ..., M-1$

(2) We can have two *equivalent* difference equations for Example 27:

4.41 provides $y[n] = \sum_{k=0}^{M} a^k x[n-k]$

4.42 provides $y[n] - ay[n-1] = x[n] - a^{M+1}x[n-M-1]$.

4.8 Structures for Discrete-Time Systems

4.8.a Block Diagram Representation of Linear Constant-Coefficients Difference Equations

Recall: A LTI system can be represented by a difference equation or, equivalently, by the system function $H(z)$.

Example 28:

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + bx[n] \quad (4.43)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - a_1 z^{-1} - a_2 z^{-2}} \quad (4.44)$$

we are interested in the representation of 4.43 or 4.44 by block diagram.

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Elements: **Assumptions**: Adders only have two inputs!

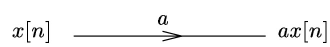


Figure 4.9: Multiplier

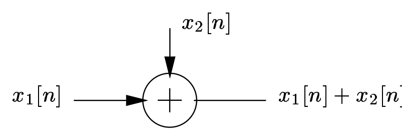


Figure 4.10: Adder

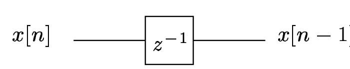
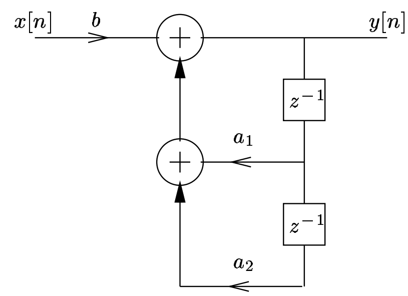


Figure 4.11: Unit Delay

The system in 4.43 can be represented by:



General Difference Equation Representation

Consider the difference equation

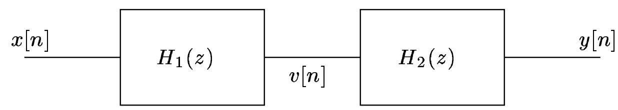
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Assume that $a_o = 1$

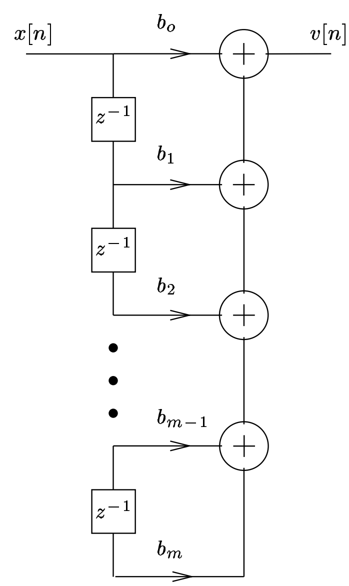
$$\begin{aligned} y[n] &= \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \\ H(z) &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \end{aligned}$$

Direct Form I Representation

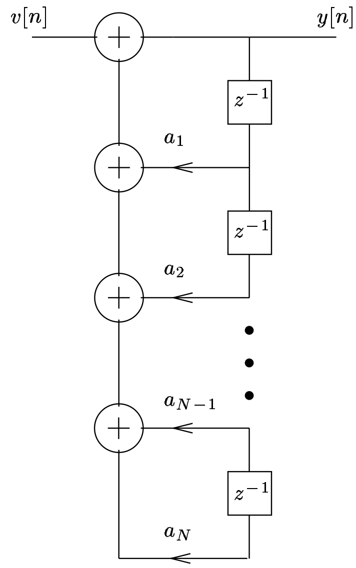
$$\begin{aligned} H(z) &= \left(\sum_{k=0}^{M} b_k z^{-k} \right) \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) \\ &\triangleq H_1(z) H_2(z) \\ &= \left(\frac{V(z)}{X(z)} \right) \cdot \left(\frac{Y(z)}{V(z)} \right) \end{aligned}$$



$$H_1(z) = \frac{V(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$
$$V(z) = X(z) \sum_{k=0}^{M} b_k z^{-k}$$



$$\begin{aligned} H_2(z) &= \frac{Y(z)}{V(z)} = \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \\ Y(z) &= Y(z) \left(\sum_{k=1}^{N} a_k z^{-k} \right) + V(z) \end{aligned}$$



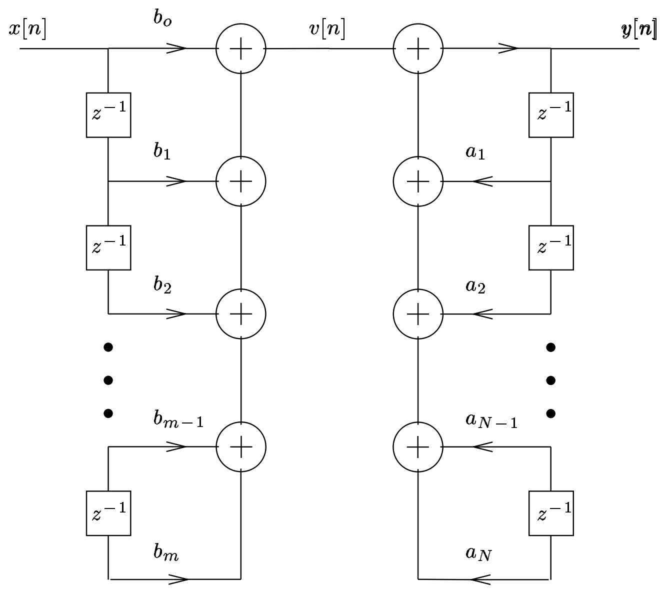
Therefore, the direct form I representation is:

Direct Form II Representation - "Canonical Form"

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{N_1} b_k z^{-k}}{1 - \sum_{k=1}^{N_1} a_k z^{-k}},$$

where $N_1 = max(M, N)$. Now let's rewrite $H(z)$ as follows:

$$\begin{aligned} H(z) &= \left(\frac{1}{1 - \sum_{k=1}^{N_1} a_k z^{-k}}\right)\left(\sum_{k=1}^{N_1} b_k z^{-k}\right) \triangleq H_2(z) \cdot H_1(z) = \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)} \\ H_2(z) &= \frac{W(z)}{X(z)} = \frac{1}{1 - \sum_{k=1}^{N_1} a_k z^{-k}} \\ X(z) &= W(z)[1 - \sum_{k=1}^{N_1} a_k z^{-k}] \end{aligned}$$



$$H_1(z) = \frac{Y(z)}{W(z)} = \sum_{k=0}^{N_1} b_k z^{-k}$$

$$Y(z) = W(z)\left(\sum_{k=0}^{N_1} b_k z^{-k}\right)$$

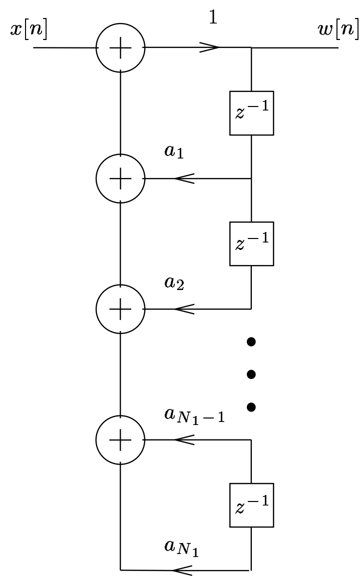
Direct form II Representation is:

Combination of Delays in Direct form II

Comments:

1. The above form requires minimum number of delays.

2. Although forms I and II are equivalent, effect of finite-precision arithmetic can be drastically different on the performance of the two forms.



4.8.b Signal Flow Graph Representation of Linear Constant-Coefficient Difference Equations

Similar to the block diagram representation with few different notations.
Example 29:

$$\begin{aligned} w_1[n] &= aw_4[n] + x[n] \\ w_2[n] &= w_1[n] \\ w_3[n] &= b_o w_2[n] + b_1 w_4[n] \\ w_4[n] &= w_3[n] \end{aligned}$$

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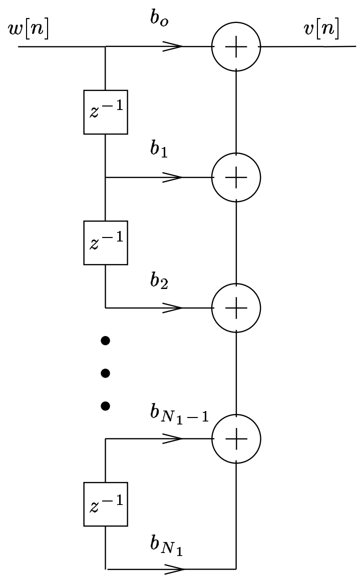
4.8.c Basic Structures for IIR Systems

General Design Considerations:

- Number of multipliers should be minimum to reduce computation time.



- Number of delays should be minimum to reduce memory size.

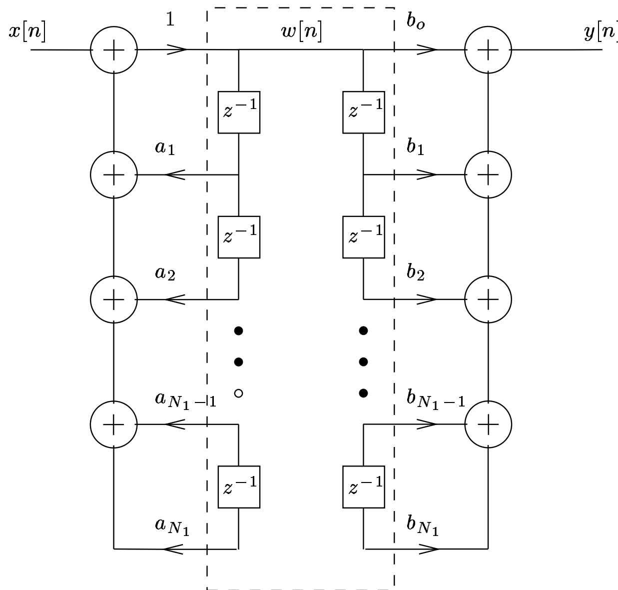
- Finite register length and finite-precision arithmetic.
  (i) These depends upon the structure used (signal flow graph).
  (ii) These effects *do not*, in general, correlate with minimum number of multipliers and/or minimum number of delays.

Direct Forms

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

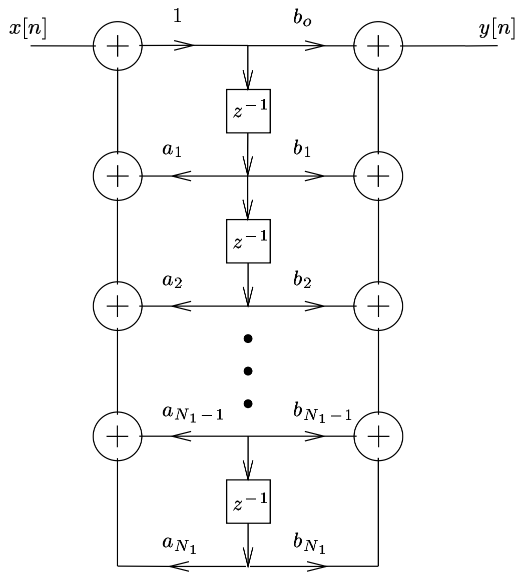
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-1}}$$

Assume $M = N$

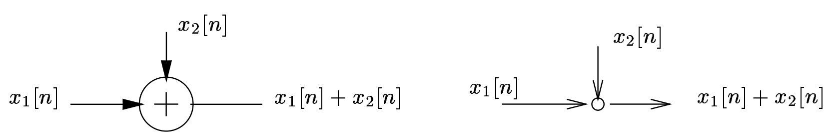


Direct Form I:

$$
\begin{aligned}
H(z) &= \left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right) \\
&= H_1(z)H_2(z) \\
&= \left(\frac{V(z)}{X(z)}\right)\left(\frac{Y(z)}{V(z)}\right)
\end{aligned}
$$

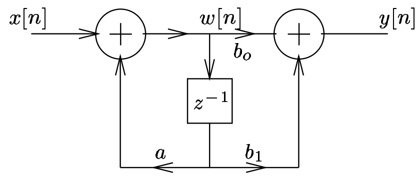


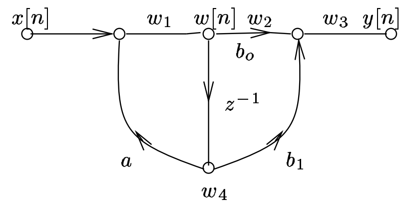
Multiplier/Gain

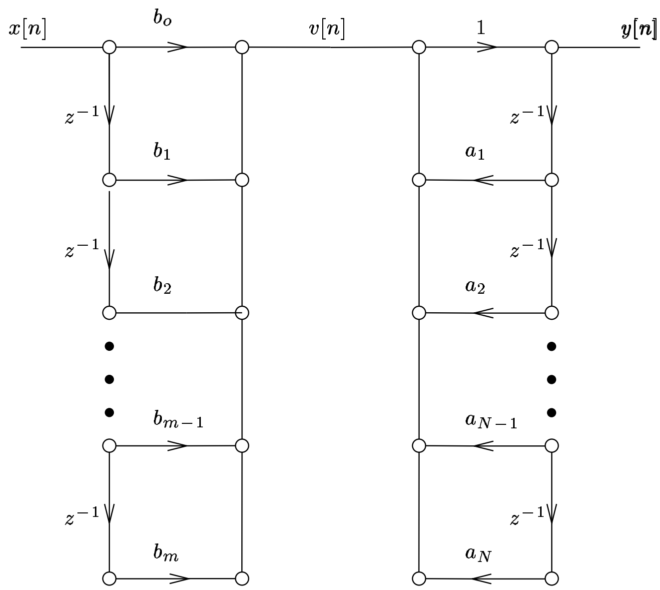


Adder

Unit Delay







Direct Form II: "Canonical Form"

Example 30:

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

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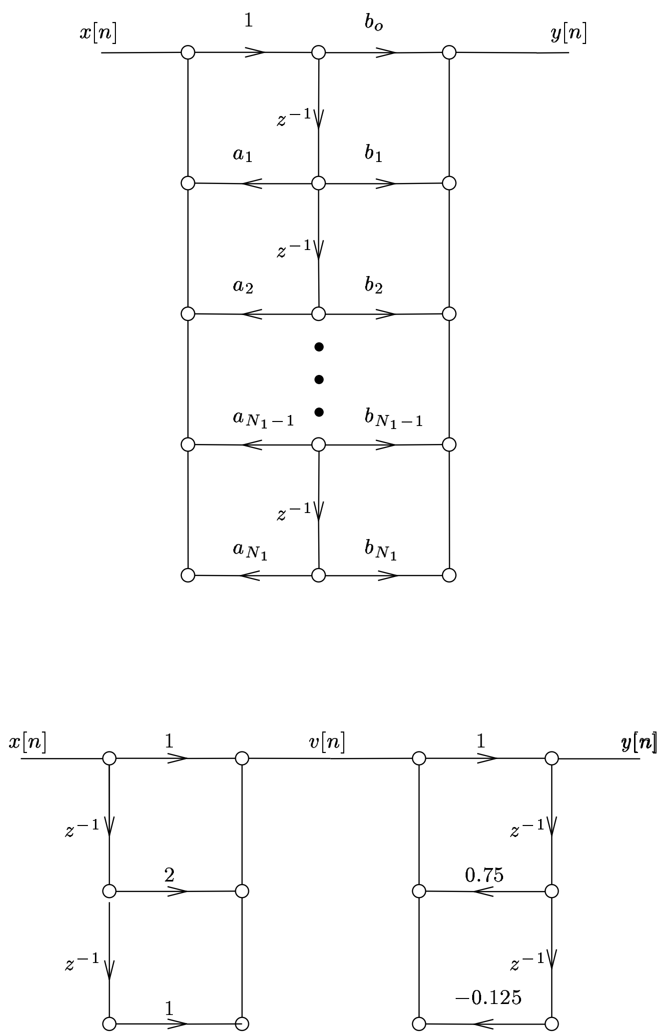

Figure 4.12: Direct Form I

4.8.d Basic Network Structures for FIR Systems

Direct Form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\equiv \sum_{k=0}^{M} h[k]x[n-k] \triangleq h[n] * x[n]$$

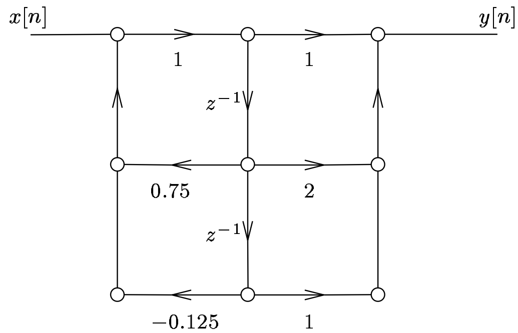

Figure 4.13: Canonical Form

where

$$h[n] = \begin{cases} b_n, & n = 0, 1, 2, ..., M \\ 0, & \text{else} \end{cases} \tag{4.45}$$