ECE 530
HW 6
Spring 08
Farag

Text 6-18 – $z = XY$

We proved in class that

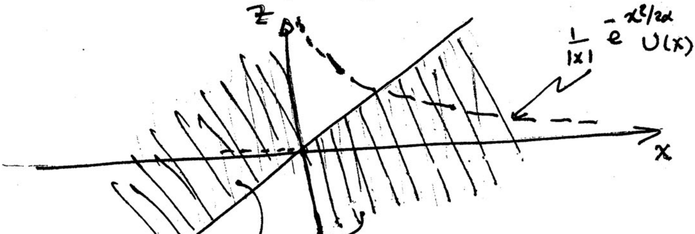
$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{XY}(x, \tfrac{z}{x})\, dx;$$

for all x.

Since X and Y are independent, we get

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) \cdot f_Y(\tfrac{z}{x})\, dx$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{1}{|x|} \frac{x e^{-x^2/2\alpha}}{\alpha^2} U(x) \cdot \frac{dx}{\pi\sqrt{1-(\frac{z}{x})^2}}, & |\frac{z}{x}| < 1 \\ 0 & \text{else where.} \end{cases}$$



region of support for $\frac{1}{\pi\sqrt{1-(\frac{z}{x})^2}}$; $\underbrace{|\frac{z}{x}| < 1}_{|x| > |z|}$

Hence, we need to integrate only for $|x| > |z|$.

$$\therefore\ f_Z(z) = \int_{|z|}^{\infty} \frac{1}{\alpha^2} e^{-x^2/2\alpha^2} \frac{1}{\pi\sqrt{1-\left(\frac{z}{x}\right)^2}}\, dx$$

$$\equiv \frac{1}{\alpha^2\pi}\int_{|z|}^{\infty} \frac{x\, e^{-x^2/2\alpha^2}}{\sqrt{x^2-z^2}}\, dx .$$

Now, let $v \triangleq \sqrt{x^2-z^2} \Rightarrow x^2 = v^2+z^2$

$$\therefore\ dv = \frac{x}{\sqrt{x^2-z^2}}\, dx$$

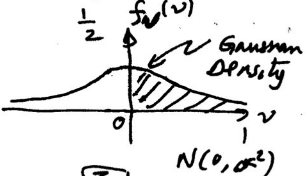
$$x\Big|_{|z|}^{\infty} \Rightarrow v\Big|_{0}^{\infty}$$

$$\therefore\ f_Z(z) = \frac{1}{\pi\alpha^2}\int_0^{\infty} e^{-(v^2+z^2)/2\alpha}\, dv$$

$$\equiv \frac{2\, e^{-z^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}} \underbrace{\int_0^{\infty} \frac{e^{-v^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}}}_{\frac{1}{2}}$$

$$= \frac{e^{-z^2/2\alpha^2}}{\sqrt{2\pi\alpha^2}}$$

$\Rightarrow Z \sim N(0, \alpha^2)$



$\boxed{Z}$

Text 6 – 19

(a) $z = \frac{x}{y}$, x, y independent

$$f_z(z) = \int_{-\infty}^{\infty} |y| \, f_{xy}(yz, y) \, dy$$

$$= \int_{-\infty}^{\infty} |y| \, f_x(yz) \cdot f_y(y) \, dy$$

$$= \int_{-\infty}^{\infty} |y| \cdot \frac{yz}{\alpha^2} e^{-y^2 z^2 / 2\alpha^2} U(yz) \cdot \frac{y}{\beta^2} e^{-y^2/2\beta^2} U(y) \, dy$$

$$= \int_0^{\infty} \frac{y^3 z}{\alpha^2 \beta^2} e^{-y^2\left[\frac{1}{2\beta^2} + \frac{z^2}{2\alpha^2}\right]} dy$$

$$= \frac{z}{\alpha^2 \beta^2} \underbrace{\int_0^{\infty} y^3 e^{-Ky^2} dy}_{\frac{1}{2K^2}} \quad ; \; z \geq 0$$

$$= \frac{z}{2\alpha^2 \beta^2 K^2} \quad ; \; K \triangleq \frac{1}{2\beta^2} + \frac{z^2}{2\alpha^2}$$

$$= \frac{2\alpha^2}{\beta^2} \frac{z}{\left(z^2 + \alpha^2/\beta^2\right)} U(z)$$

— Q.E.D

(b) $P(x \le ky) = P(x/y \le k) = P(z \le k);\ z = x/y$

$$= \int_0^k \frac{2\alpha^2}{\beta^2} \cdot \frac{z}{z^2 + \frac{\alpha^2}{\beta^2}}\, dz$$

$$= \frac{k^2}{k^2 + \alpha^2/\beta^2}.$$

Text 6-20

1. The density of $2x$ equals $\frac{1}{2} f_x(\frac{x}{2})$. Hence, if $z = 2x + y$, then

$$f_z(z) = \int_0^z \frac{\alpha}{2} e^{-\alpha x/2} \beta e^{-\beta(z-x)} dx = \frac{\alpha\beta}{\alpha - 2\beta}(e^{-\beta z} - e^{-\alpha z/2})U(z)$$

2. The density of y equals $f_y(-y)$. Hence, if $z = x - y$, then

$$f_z(z) = f_x(z) * f_y(-z)$$

$$= \alpha\beta \begin{cases} \displaystyle\int_z^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha z} & z > 0 \\ \displaystyle\int_0^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{\beta z} & z < 0 \end{cases}$$

3. $z = x/y \qquad w = y \qquad J = 1/y$

$$f_z(z) = \alpha\beta \int_0^\infty w e^{-\alpha z w} e^{-\beta w} dw = \frac{\alpha\beta}{(\alpha z + \beta)^2} U(z)$$

4. $z = \max(x, y) \qquad F_z(z) = F_{xy}(z, z) = F_x(z)F_y(z)$

$$f_z(z) = f_x(z)F_y(z) + f_y(z)F_x(z)$$

$$= \left[\alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z})\right] U(z)$$

5. $z = \min(x, y) \qquad F_z(z) = F_x(z) + F_y(z) - F_x(z)F_y(z)$

$$f_z(z) = f_x(z)[1 - F_y(z)] + f_y(z)[1 - F_x(z)] = (\alpha + \beta)e^{-(\alpha+\beta)z} U(z)$$

Text 6-39

$$z = x + a\cos y \triangleq x + w$$

$$w \triangleq a\cos y$$

$$\therefore\ f_W(w) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2-w^2}} & |w| < a \\ 0 & |w| > a \end{cases}$$

$$f_Z(z) = f_X(x) * f_W(w)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

$$\therefore\ f_Z(z) = \int_{-\infty}^{\infty} f_W(w)\, f_X(z-w)\, dw$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{1}{\pi\sqrt{a^2-w^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-w)^2/2\sigma^2}\, dw & |w| < a \\ 0 \ ; & \text{else.} \end{cases}$$

$$= \int_{-a}^{a} \frac{1}{\pi\sqrt{a^2-w^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(z-w)^2/2\sigma^2}\, dw$$

$$= \frac{1}{\pi\sigma\sqrt{2\pi}} \int_0^{\pi} e^{-(z-a\cos y)^2/2\sigma^2}\, dy.$$

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